

NON-ARCHIMEDEAN PERIODS FOR LOG CALABI-YAU SURFACES

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ABSTRACT. Non-archimedean geometry has proven to be a very powerful technique to study classical problems in algebraic geometry. Whilst non-archimedean analytic spaces can be very complicated, there exists a canonical combinatorial subspace, called the essential skeleton, which retains a lot of geometric information. In this thesis, we explore how much information the skeleton of a non-archimedean analytic space captures. Our methods are directly inspired by mirror symmetry and the SYZ conjecture, in which non-archimedean geometry plays a fascinating role as first observed by Kontsevich and Soibelman [KS01].

Using the non-archimedean SYZ fibration, we study the notion of non-archimedean periods on the essential skeleton as introduced by Kontsevich and Soibelman. We prove that the non-archimedean period map recovers the analytic periods for log Calabi-Yau surfaces, verifying a conjecture of Kontsevich and Soibelman [KS06] in this case.

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1. INTRODUCTION

Let X be a complex manifold and ω a holomorphic differential k -form on X . A *period* with respect to ω , is the integral $\int_{\alpha} \omega$ over a singular k -cycle α on X . These integrals contain valuable geometric information about X . The main result of this paper shows that one can compute these integrals when X is a log Calabi-Yau surface using novel ideas from non-archimedean geometry.

1.1. Main result. Let (Y, D) be a pair with Y a smooth projective surface over \mathbb{C} and $D \in |-K_Y|$ either an irreducible rational nodal curve or a cycle of $n \geq 2$ smooth rational curves. We call such a pair a *Looijenga pair*. We say $U = Y \setminus D$ is a *log Calabi-Yau surface* which has a distinguished volume form Ω (unique up to scaling). We consider the unique volume form Ω on U such that $\int_{\gamma_0} \Omega = (2\pi i)^2$ where γ_0 is the homology class of a 2-torus inside U . We say the pair (Y, D) is *generic* if it contains no smooth rational curves of self-intersection -2 disjoint from D ; similarly we say U is *generic* if it contains no smooth rational curves of self intersection -2 .

Let $\pi : U \rightarrow \Delta^*$ be a family of generic log Calabi-Yau surfaces over the punctured disk $\Delta^* \subseteq \mathbb{C}^*$. Denote $U_t = \pi^{-1}(t)$ and let Ω_t be the normalised volume form on U_t . Then one can define the *analytic period map*

$$P : H_2(U_t, \mathbb{Z}) \longrightarrow K^\times$$

$$\gamma_t \longmapsto \exp \left(2\pi i \frac{\int_{\gamma_t} \Omega_t}{\int_{\gamma_0} \Omega_t} \right)$$

Consider the field $K = \mathbb{C}((t))$ of Laurent series, endowed with the t -adic absolute value given by the order of vanishing at $t = 0$. Then K is a complete non-archimedean field. The family U can be viewed as a variety over K and we can associate a topological space, U^{an} called the *Berkovich analytification of U* . This is a compactification of the space of real valuations on the function field of U and there exists a map of topological spaces

$$\pi : U^{\text{an}} \longrightarrow \text{Sk}(U)$$

onto a canonical piecewise linear subspace $\text{Sk}(U) \subseteq U^{\text{an}}$, called the *essential skeleton*. Outside of a discriminant locus $\Gamma \subseteq \text{Sk}(U)$ of codimension 2, π is an affinoid torus fibration; this is an analogue of a smooth real torus fibration. We refer to this as a *non-archimedean SYZ fibration*. We review this in §2. The subspace $\text{Sk}(U)$ is homeomorphic to \mathbb{R}^2 . The fibration induces an integral affine structure with singularities on $\text{Sk}(U)$; locally the integral affine functions are absolute values of invertible analytic functions on the fibers of the fibration. We construct a fibration such that $\text{Sk}(U)^{\text{sm}}$ is an integral affine manifold with only focus-focus singularities in §3. By taking a suitable quotient of $\pi_*(\mathcal{O}_{U^{\text{an}}}^\times)$, Kontsevich and Soibelman [KS06] consider a refinement of the integral affine structure which they call a K -affine structure. We denote this quotient by Aff_K and it fits into a short exact sequence

$$0 \rightarrow K^\times \rightarrow \text{Aff}_K \rightarrow \check{\Lambda} \rightarrow 0$$

where $\check{\Lambda}$ is the sheaf of integral cotangent vectors on $\text{Sk}(U)^{\text{sm}}$. The K -affine structure Aff_K defines an element of $\text{Ext}^1(\check{\Lambda}, K^\times) \cong H^1(\text{Sk}(U)^{\text{sm}}, \Lambda \otimes K^\times)$, where Λ is the sheaf of integral tangent vectors on $\text{Sk}(U)^{\text{sm}}$. Then the *non-archimedean period map* is defined as the map

$$\bar{P} : H_1(\text{Sk}(U)^{\text{sm}}, \check{\Lambda}) \longrightarrow K^\times$$

given by the natural pairing between homology and cohomology. We then prove the following result.

Theorem 1.1. Let $\pi : U \rightarrow \Delta^*$ be a family of generic log Calabi-Yau surfaces. Then there exists a non-archimedean SYZ fibration $\rho : U^{\text{an}} \rightarrow \text{Sk}(U)$ such that the non-archimedean period map \bar{P} factors through the analytic period map P of U .

Remark 1.2. In fact, we prove a more general statement. Given a Looijenga pair (Y, D) such that $U = Y \setminus D$ is a generic log Calabi-Yau surface over K , we prove that the non-archimedean period map P factors through the period map associated to the Looijenga pair (§3). When X is defined over \mathbb{C} , one can show that this map agrees with the analytic period map.

Remark 1.3. The fibration ρ depends on a choice of model which can only be constructed in the case of a generic log Calabi-Yau surface. By an analytic continuation argument in §6, we can prove Theorem 1.1 for all log Calabi-Yau surfaces.

This proves the following conjecture of Kontsevich-Soibelman [KS06] for all log Calabi-Yau surfaces.

Conjecture 1.4 ([KS06] Conjecture 10). Given a family of n -dimensional (log) Calabi-Yau variety X over Δ^* , the analytic period map $P : H_n(X_t, \mathbb{Z}) \rightarrow K^\times$ factors through the non-archimedean period map \bar{P} .

By Carlson's theory of extensions of mixed Hodge structures [Car85], we know the mixed Hodge structure $H^2(U)$ of the open surface U is classified by the analytic period map. In particular, Theorem 1.1 implies that the K -affine structure recovers the isomorphism type of U by a simple application of the Torelli theorem for log Calabi-Yau surfaces [Fri16, GHK15b]. We review this in §6.

1.2. Motivation. The use of non-archimedean geometry techniques is motivated by mirror symmetry. Let $X \xrightarrow{\pi} \Delta^*$ be a family of n -dimensional (log) Calabi-Yau varieties over the punctured disk $\Delta^* \subseteq \mathbb{C}^*$. We further assume the family is maximally degenerate *i.e.* the monodromy acting on the degree n cohomology of the general fiber is maximally unipotent. Then the Strominger-Yau-Zaslow (SYZ) conjecture [SYZ96] roughly states that the general fiber $X_t = \pi^{-1}(t)$, admits a fibration $\rho : X_t \rightarrow B$, whose base B is a real n -dimensional topological manifold and the fibers are special Lagrangian tori away from a discriminant locus Γ of codimension ≥ 2 in B . The SYZ fibration ρ endows B with a singular integral affine structure *i.e.* in the complement of the discriminant locus of ρ , the transition functions between charts of B are affine transformations in $\text{GL}_n(\mathbb{Z}) \rtimes \mathbb{R}$. The mirror partner \check{X}_t of X_t can then be constructed by dualising the torus fibration over the smooth locus and deforming the dual fibration by so-called quantum corrections.

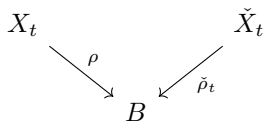


FIGURE 1. Strominger-Yau-Zaslow conjecture

The construction of the special Lagrangian fibration is very difficult and has been done in very few cases. A fundamental insight of Kontsevich and Soibelman [KS06] is that one should be able to construct a close analogue of the fibration in the world of non-archimedean geometry. Unlike its archimedean version, the non-archimedean SYZ fibration has been constructed in general [NXY19].

We sketch the idea below and give further details in §2. It is important to note that the non-archimedean version can realise the original goal of constructing a mirror by non-archimedean GAGA and algebrisation techniques.

Given a variety over a complete non-archimedean field K , the *Berkovich analytification* X^{an} of X [Ber90] is a locally ringed space which is a compactification of the space of real valuations on $K(X)$ that extend the t -valuation on K . It satisfies several nice topological properties: X^{an} a Hausdorff, connected, locally contractible topological space, which is compact whenever we assume X/K is proper. Let $K = \mathbb{C}((t))$, equipped with the t -adic absolute value $|\cdot|_K$. Then given a (log) Calabi-Yau variety over K , there exists a piecewise linear subspace $\text{Sk}(X) \subseteq X^{\text{an}}$, called the *essential skeleton*, and a strong deformation retraction $\rho : X^{\text{an}} \rightarrow \text{Sk}(X)$. The construction of the retraction depends on a choice of degeneration of X and in [NXY19], the authors prove there is a minimal choice (but not necessarily unique) which gives rise to an affinoid torus fibration away from a discriminant locus in $\text{Sk}(X)$ of codimension ≥ 2 . The fibration induces a singular integral affine structure on $\text{Sk}(X)$ given by the pushforward of the sheaf of invertible analytic functions via ρ . With some modification, this can be shown to agree with the integral affine structure arising from classical SYZ fibrations in certain cases [MPS23]. Choosing a different model results in a different integral affine structure.

Following ideas in [KS06], we show that the essential skeleton $\text{Sk}(X)$ and the K -affine structure contain important information about X . This is in the spirit of the reconstruction problem in mirror symmetry which says that one should be able to recover the original variety from the base of the SYZ fibration and certain structures on it, such as the integral affine structure.

1.3. Relations to other works. To the best of our knowledge, the approach of computing periods using non-archimedean techniques in the context of mirror symmetry has not been explored other than the suggestions of [KS06]. There has been work in understanding the relationship between tropical geometry and periods. For example, it is known that minus the valuation of the j -invariant of a degeneration elliptic curve over a non-archimedean field coincides with the cycle length of the tropical elliptic curve obtained by tropicalisation [Mar09, KMM09]. The period computation in §6.1 is a generalisation of this result for the case of abelian varieties.

The leading term of period integrals of families of varieties has been also computed using tropical geometry: the case of Riemann surfaces was done in [Iwa10], toric Calabi-Yau hypersurfaces in [AGIS20] and toric hypersurfaces in [?].

Hosono's conjecture [Hos06] predicts that periods on the mirror equal the pairing of certain explicit hypergeometric series with the Chern classes of vector bundles, for Batyrev mirror pairs of toric Calabi-Yau hypersurfaces. By studying the asymptotics of Hosono's conjecture in the large complex structure limit, one arrives at the mirror symmetry $\hat{\Gamma}$ -conjecture [Iri23]. Abouzaid-Ganatra-Iritani-Sheridan [AGIS20] use the tropicalisation map, which acts as a limiting SYZ fibration, to compute the leading term of these period integrals for toric Calabi-Yau hypersurfaces. It would be interesting to explore how the non-archimedean method could aid in the computation.

Ruddat-Siebert [RS20] have computed periods of toric degenerations arising from the Gross-Siebert program. In particular, they compute the period integrals of holomorphic volume forms over n -dimensional cycles corresponding to tropical one-cycles on the central fiber of the degeneration. Their formalism is similar in nature to the non-archimedean, except they do not construct an exact (non-archimedean) SYZ fibration. Moreover, the Gross-Siebert program [GS11, GS21] only considers toric degenerations whereas we allow any reasonable degeneration arising from birational geometry. Theorem 1.1 is the first test case for Conjecture 1.4, which goes beyond the case of toric degenerations.

1.4. Outline. In §2, we review several preliminaries we will need throughout the paper. Here we will introduce all the necessary theory from Berkovich analytic geometry. In §3, we introduce log Calabi-Yau surfaces in more detail and construct the degeneration and fibration needed for the proof of our main results. We give an explicit description of the polyhedral structure on the essential skeleton and its singular integral affine structure. We also introduce the period map for a Looijenga pair. In §4, we define the group of tropical cycles $H_1(\mathrm{Sk}(U)^{\mathrm{sm}}, \check{\Lambda})$ which is the domain of the non-archimedean period map. We prove a tropical correspondence result with the period domain for a Looijenga pair. In §5, we define K -affine structures and construct one which is a lift of the integral affine structure on the essential skeleton. In this section, we prove a Torelli theorem saying that K -affine structures on real flat tori parameterise maximally degenerate abelian varieties over K . In §6, we define the non-archimedean period and compute the periods for uniformised abelian varieties and log Calabi-Yau surfaces. This will complete the proof of Theorem 1.1. In the appendix, we give an explicit computation of the non-archimedean period map for the case of non-toric DP6 and DP7.

2. PRELIMINARIES

Throughout this thesis, we denote by $K = \mathbb{C}((t))$ the field of Laurent series over \mathbb{C} ; it has the structure of a complete discrete valuation field with uniformiser t . Let $R = \mathbb{C}[[t]]$ denote its ring of integers and $k = \mathbb{C}$, its residue field. We denote the discrete valuation by $\text{val}_K : K^\times \rightarrow \mathbb{Z}$ and define an absolute value $|a|_K = \exp(-\text{val}_K(a))$. We will always take a K -variety to mean a normal and geometrically irreducible K -scheme.

In this chapter, we will give several preliminary definitions and results that will be needed in this thesis. We give an overview of the chapter first. In §2.1, we will define an R -model of a K -variety and study a specific class of models called *simple normal crossings (snc) models*. We will also introduce a relative notion of an snc model, called an *snc log model*. In §2.2, we introduce the Berkovich analytification of a K -variety and describe its connection to birational geometry and formal schemes. In §2.4, we define the combinatorial notion of a skeleton which encodes the intersection patterns of the irreducible components in an snc (log) model. We then prove the Berkovich analytification of a K -variety retracts onto the skeleton of an snc (log) model (2.19) and give a local description (2.20) of the retraction. Having defined the skeleton associated to an snc log model, we will introduce the essential skeleton associated to an snc pair in §2.5 and give a criterion for when they coincide. In §2.6, we define the notion of integral affine structures and in §2.7 introduce the tropicalisation map and an affinoid torus fibration. We end by discussing how an affinoid torus fibration induces an integral affine structure on the skeleton of an snc (log) model and define a *non-archimedean SYZ fibration*.

2.1. Models.

Definition 2.1. Let X be a normal integral K -scheme of finite type.

- (1) An R -model of X is a proper flat R -scheme \mathcal{X} of finite type endowed with an isomorphism $\mathcal{X}_K \rightarrow X$;
- (2) If \mathcal{X} and \mathcal{Y} are R -models of X , then a morphism of R -models of X is a morphism of R -schemes $h : \mathcal{X} \rightarrow \mathcal{Y}$ such that $h_K : \mathcal{X}_K \rightarrow \mathcal{Y}_K$ is an isomorphism that commutes with the isomorphisms $\mathcal{X}_K \rightarrow X$ and $\mathcal{Y}_K \rightarrow X$;
- (3) A *simple normal crossings (snc) model* is a regular R -model \mathcal{X} whose special fibre \mathcal{X}_k is a divisor with simple normal crossings singularities.

We record the following definition for use later.

Definition 2.2. Let \mathcal{X} be a Noetherian scheme and D an effective divisor on \mathcal{X} , with prime components D_i for $i \in I$. A *stratum* of D is a connected component of the scheme-theoretic intersection $D_J = \bigcap_{j \in J} D_j$ for some $J \neq \emptyset \subseteq I$. An *open stratum* is a stratum S minus the union of the prime components of D which do not contain S . Given a stratum S of D , let S° denote the open stratum of S .

Definition 2.3. An *snc pair* $(\mathcal{X}, \mathcal{H})$ consists of a proper regular flat scheme \mathcal{X} over R and $\mathcal{H} = \sum_{i \in I} \mathcal{D}_i$ a sum of effective Cartier divisors on \mathcal{X} such that

- (i) each \mathcal{D}_i has irreducible support;
- (ii) the divisor $\mathcal{X}_k + \mathcal{H}$ is snc.

Given an snc pair $(\mathcal{X}, \mathcal{H})$, we will be solely interested in the case when the generic fibre of $\mathcal{X} \setminus \mathcal{H}$ is a log Calabi-Yau surface. Nevertheless, we make a more general definition of a model relative to an snc divisor:

Definition 2.4. Let X be an algebraic variety over K . An *snc log-model* of X is an snc pair $(\mathcal{X}, \mathcal{H})$ together with an isomorphism $X \cong (\mathcal{X} \setminus \mathcal{H})_K$.

Example 2.5. Let $K' = \mathbb{C}((t^{\frac{1}{2}}))$, a degree 2 extension of K and $R' = \mathbb{C}[[t^{\frac{1}{2}}]]$. We have that $K' \cong K$ as discretely valued fields. Then take $X = \mathbb{G}_m$ over K' and consider the model of X with special fibre given by 2 copies of \mathbb{P}^1 and general fibre $\mathcal{X}_{K'} \cong \mathbb{P}^1$:

$$\mathcal{X} = \text{Proj} \left(\frac{R'[X_0, X_1, X_2]}{(X_0 X_1 - t X_2^2)} \right).$$

Let \mathcal{H} be the scheme-theoretic closure of $0, \infty \in \mathcal{X}_{K'}$ in \mathcal{X} . Then $(\mathcal{X}, \mathcal{H})$ is an snc log model of X .

The existence of an snc log model of X imposes a strict condition on X :

Lemma 2.6. An snc log model of X exists if and only if X is smooth.

Proof. First suppose X is smooth. By Hironaka's resolution of singularities [Hir64], there exists an snc compactification Y of X *i.e.* a dense open embedding of X into a smooth proper K -scheme Y such that the boundary $D = Y \setminus X$ (with its induced reduced structure) is an snc divisor. Then by Nagata compactification [Sta24, Section 0F3T], there is a proper and flat R -scheme \mathcal{Y} whose generic fibre is isomorphic to Y . Let \mathcal{D} be the scheme-theoretic closure of D inside \mathcal{Y} . If $\mathcal{Y}_k + \mathcal{D}$ is snc, then we are done. Otherwise, by another application of resolution of singularities, there is a proper birational morphism $h : \mathcal{Y}' \rightarrow \mathcal{Y}$ such that \mathcal{Y}' is regular, $h_K : \mathcal{Y}'_K \rightarrow \mathcal{Y}_K$ is an isomorphism and $h^* \mathcal{D} + \mathcal{Y}'_k$ is snc. Then the pair $(\mathcal{Y}', h^* \mathcal{D})$ is an snc log model of X .

Suppose an snc log model $(\mathcal{X}, \mathcal{H})$ of X exists. By definition, \mathcal{X}_K is regular and K is a perfect field so \mathcal{X}_K is smooth. Thus $X \subset \mathcal{X}_K$ is smooth. \square

Let $(\mathcal{X}, \mathcal{H})$ be an snc pair and consider the finite set $\{\mathcal{D}_i\}_{i \in I^\vee}$ of irreducible components of the special fibre $\mathcal{X}_k^{\text{red}}$ with reduced scheme structure. For every $i \in I^\vee$, let N_i denote the multiplicity of \mathcal{D}_i in \mathcal{X}_k . We refer to the divisors \mathcal{D}_i indexed by $i \in I^\vee$ as *vertical divisors*. Likewise, the divisors indexed by $i \in I^{\text{h}}$ are called *horizontal divisors*. This terminology matches up with the usual definitions of horizontal and vertical divisors in the literature [GRW16]. For every non-empty set $I \subseteq I^\vee \cup I^{\text{h}}$, we write $\mathcal{D}_I = \cap_{i \in I} \mathcal{D}_i$.

2.2. Berkovich spaces. The theory of Berkovich spaces provides a robust framework for making sense of analytic geometry over non-archimedean fields. We will not be concerned with the full theory in this thesis but rather with a particular construction called *analytification* which associates to each variety, a space of valuations. These spaces satisfy several nice topological properties which encode the geometry of the underlying algebraic variety. We recall the main definitions and constructions in this section. Let X be a K -variety.

Definition 2.7 ([Ber90]). The *Berkovich analytification* of X is set-theoretically given by pairs $(\xi_x, |\cdot|_x)$ where:

- (i) ξ_x is a scheme-theoretic point of X ;
- (ii) $|\cdot|_x : \kappa(x) \rightarrow \mathbb{R}_{\geq 0}$ is an absolute value on the residue field $\kappa(x)$ extending the t -adic absolute value $|\cdot|_K$ on K .

It comes equipped with the weakest topology such that:

- (1) the forgetful map $\iota : X^{\text{an}} \rightarrow X$ is continuous;

(2) for any Zariski open $U \subseteq X$ and $f \in \mathcal{O}_X(U)$, the evaluation map

$$\begin{aligned} |f| : \iota^{-1}(U) &\longrightarrow \mathbb{R}_{\geq 0} \\ (\xi_x, |\cdot|_x) &\longmapsto |f|_x \end{aligned}$$

is continuous.

Example 2.8. When $X = \text{Spec } A$ is affine, X^{an} is given by the set of multiplicative seminorms on A that extend the absolute value $|\cdot|_K$ on K .

The topological space X^{an} is equipped with a sheaf of analytic functions which makes $\iota : X^{\text{an}} \rightarrow X$ a morphism of locally ringed spaces. The topological space X^{an} is Hausdorff, and it is compact if and only if X/K is proper. The subset of birational points $X^{\text{bir}} \subseteq X^{\text{an}}$ is defined as $\iota^{-1}(\eta_X)$ where η_X is the generic point of X . These points have the form $(\eta_X, |\cdot|)$ where $|\cdot| : K(X) \rightarrow \mathbb{R}_{\geq 0}$ is an absolute value on the function field $K(X)$ of X extending the absolute value on K .

The following example illustrates the connections between Berkovich analytic geometry and birational geometry when working over discrete valuation fields. In particular, one can single out a large class of birational points using the theory of models.

Example 2.9. Let \mathcal{X} be a R -model of X and E an irreducible component of \mathcal{X}_k . Denote the generic point of E by ξ_E . Then $\mathcal{O}_{\mathcal{X}, \xi_E}$ is a discrete valuation ring whose valuation v_E is given by the order of vanishing along E and its fraction field is $K(X)$. The normalised valuation $\frac{1}{N_E}v_E$, where N_E is the multiplicity of E in \mathcal{X}_k , determines an absolute value on $K(X)$ which extends the absolute value on K . This defines a birational point x of X^{an} and is called a *divisorial point* of X^{an} .

More generally, given an snc model \mathcal{X} of X , let the tuple (E_1, \dots, E_r) denote distinct irreducible components of \mathcal{X}_k such that $E = \bigcap_i E_i \neq \emptyset$. Let ξ_E denote the generic point of a connected component of E . Then there is a regular system of local parameters $z_1, \dots, z_r \in \mathcal{O}_{\mathcal{X}, \xi_E}$, positive integers N_1, \dots, N_r and a unit $u \in \mathcal{O}_{\mathcal{X}, \xi_E}^\times$ such that

$$t = uz_1^{N_1} \dots z_r^{N_r}$$

where locally at ξ_E , the local equation $z_i = 0$ defines E_i and N_i is the multiplicity of E_i in \mathcal{X}_k . For each choice of tuple $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}_{\geq 0}^r$ such that $\sum_i \alpha_i N_i = 1$, there exists a unique valuation [MN15, Proposition 2.4.4]

$$v_{E, \alpha} : \mathcal{O}_{\mathcal{X}, \xi_E} \setminus \{0\} \longrightarrow \mathbb{R}$$

such that $v_{E, \alpha}(z_i) = \alpha_i$ for all i . This extends the valuation on K since

$$v_{E, \alpha}(t) = v(uz_1^{N_1} \dots z_r^{N_r}) = \sum_i N_i v_{E, \alpha}(z_i) = \sum_i \alpha_i N_i = 1.$$

The valuation $v_{E, \alpha}$ thus defines a birational point on X^{an} that we call a *monomial point* of X^{an} . We have the inclusions

$$X^{\text{div}} \subseteq X^{\text{mon}} \subseteq X^{\text{bir}} \subseteq X^{\text{an}},$$

where X^{div} and X^{mon} are the subsets of divisorial and monomial points respectively. By [MN15, Proposition 2.4.9], the space of divisorial points X^{div} is dense in X^{an} .

2.3. Connection to formal schemes.

Notation 2.10. If \mathcal{X} is a Noetherian R -scheme and C is a subscheme of \mathcal{X} , we let $\widehat{\mathcal{X}}_C$ denote the formal completion of \mathcal{X} along C . If C is equal to the special fibre, we will simply write $\widehat{\mathcal{X}}$.

Definition 2.11. We define the following subring $R\{x_1, \dots, x_n\} \subseteq R[[x_1, \dots, x_n]]$:

$$R\{x_1, \dots, x_n\} := \left\{ f = \sum_{I \in \mathbb{Z}_{\geq 0}^n} c_I x^I : c_I \in R, |c_I|_K \rightarrow 0 \text{ as } |I| \rightarrow \infty \right\}$$

where $x^I = x_1^{I_1} \dots x_n^{I_n}$.

Suppose \mathcal{X} is a scheme of finite type over R . Then $\widehat{\mathcal{X}}_C$ is formally of finite type. This means it has a finite cover by open formal subschemes of the form $\mathrm{Spf}(A)$ where A is a quotient of a topological R -algebra of the form $R\{x_1, \dots, x_m\}[[y_1, \dots, y_n]]$.

The formal completion $\widehat{\mathcal{X}}$ of an R -model \mathcal{X} of X along its special fibre is a flat separated formal R -scheme of finite type, and we can consider its (analytic) generic fibre $\widehat{\mathcal{X}}_\eta$ in the category of Berkovich analytic spaces [Ber94, §1]. This is a compact analytic domain of X^{an} . We have the following characterisation of the points in the generic fibre: a point $x = (x, |\cdot|_x) \in X^{\mathrm{an}}$ belongs to $\widehat{\mathcal{X}}_\eta$ if and only if the morphism

$$\mathrm{Spec} \mathcal{H}(x) \longrightarrow X$$

extends to a morphism

$$\mathrm{Spec} \mathcal{H}(x)^\circ \longrightarrow \mathcal{X}$$

where $\mathcal{H}(x)$ is the completion of the residue field $\kappa(x)$ with respect to $|\cdot|_x$ and $\mathcal{H}(x)^\circ$ denotes its valuation ring. If this occurs, the image of the closed point of $\mathrm{Spec} \mathcal{H}(x)^\circ$ is called the *centre* (or *reduction*) of x on \mathcal{X} and we say that x admits a centre on \mathcal{X} .

If X is proper over K then $X^{\mathrm{an}} = \widehat{\mathcal{X}}_\eta$ by the valuative criterion of properness. There is an anti-continuous reduction map

$$\mathrm{red}_X : \widehat{\mathcal{X}}_\eta \longrightarrow \mathcal{X}_k$$

that sends a point x to the centre of x on \mathcal{X} .

Example 2.12. When $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$ is affine, then

$$\widehat{\mathcal{X}}_\eta = \{x \in X^{\mathrm{an}} : |f(x)| \leq 1 \text{ for all } f \in \mathcal{A}\}.$$

2.4. Skeletons. In this section, we will introduce the combinatorial gadget needed to understand the base of the non-archimedean SYZ fibration.

Definition 2.13. The *dual intersection complex* of a (pure dimensional) simple normal crossings model \mathcal{X} is the simplicial complex $D(\mathcal{X}_k)$ whose vertices are in correspondence with the irreducible components of \mathcal{X}_k and whose r -dimensional cells correspond to strata of codimension $r + 1$, where this correspondence is inclusion-reversing.

The dual intersection complex is a compact cell complex which reflects the geometry of the generic fibre. For example, if the generic fibre is rationally connected, then the dual complex of the special fibre is contractible [dFKX17]. We have the following relative version of the dual intersection complex:

Definition 2.14. We define the following simplicial cone complex of an snc pair $(\mathcal{X}, \mathcal{H})$:

$$C_{(\mathcal{X}, \mathcal{H})} := \left\{ \sum_{i \in I^v \cup I^h} a_i \langle \mathcal{D}_i \rangle \mid a_i \geq 0, \bigcap_{i: a_i > 0} \mathcal{D}_i \neq \emptyset \right\} \subset \mathbb{R}_{\geq 0}^{I^v \cup I^h}$$

and the polyhedron:

$$\Sigma_{(\mathcal{X}, \mathcal{H})} := C_{(\mathcal{X}, \mathcal{H})} \cap \left\{ \sum_{i \in I^v} N_i a_i = 1 \right\}.$$

The polyhedron $\Sigma_{(\mathcal{X}, \mathcal{H})}$ will be called the *skeleton* of the pair $(\mathcal{X}, \mathcal{H})$. Note that when $\mathcal{H} = 0$, we recover the definition above of the dual intersection complex of \mathcal{X} . In this case, we will simply write $\Sigma_{\mathcal{X}} := D(\mathcal{X}_k)$.

Remark 2.15. The complexes $C_{(\mathcal{X}, \mathcal{H})}$ and $\Sigma_{(\mathcal{X}, \mathcal{H})}$ are called Clemens cones and polytopes respectively in [KS06, Yu16].

We present an alternative combinatorial description similar to Definition 2.13 when \mathcal{X}_k is reduced. This readily follows from the definition above.

Lemma 2.16. Let $(\mathcal{X}, \mathcal{H})$ be an snc pair and suppose that $N_i = 1$ for all i . Given $I \subset I^v$, define

$$\Delta_I := \left\{ \sum_{i \in I} a_i = 1 \right\} \subset \mathbb{R}_{\geq 0}^{I^v \cup I^h}.$$

Then the skeleton $\Sigma_{(\mathcal{X}, \mathcal{H})}$ of the pair $(\mathcal{X}, \mathcal{H})$ is the simplicial subcomplex of $\mathbb{R}^{I^v \cup I^h}$ such that:

- (i) Δ_I is a face of $\Sigma_{(\mathcal{X}, \mathcal{H})}$ if and only if $\mathcal{D}_I \neq \emptyset$ for $I \subset I^v$
- (ii) $\Delta_I \times \mathbb{R}_{\geq 0}^{|J|-1}$ is a face of $\Sigma_{(\mathcal{X}, \mathcal{H})}$ if and only if $\mathcal{D}_I \cap \mathcal{D}_J \neq \emptyset$ for $I \subset I^v$ and $J \subset I^h$ with $J \neq \emptyset$. Note we allow $I = \emptyset$ in this case.

We define the following extended simplicial cone complexes which we will use when we study the tropicalisation of algebraic cycles in log Calabi-Yau surfaces in §4.24:

$$\begin{aligned} \overline{C}_{(\mathcal{X}, \mathcal{H})} &:= \overline{\left\{ \sum_{i \in I^v \cup I^h} a_i \langle \mathcal{D}_i \rangle \mid a_i \geq 0, \bigcap_{i: a_i > 0} \mathcal{D}_i \neq \emptyset \right\}} \subset \overline{\mathbb{R}}_{\geq 0}^{I^v \cup I^h} = [0, \infty]^{I^v \cup I^h} \\ \overline{\Sigma}_{(\mathcal{X}, \mathcal{H})} &:= \overline{C}_{(\mathcal{X}, \mathcal{H})} \cap \left\{ \sum_{i \in I^v} N_i a_i = 1 \right\} \subset \overline{\mathbb{R}}_{\geq 0}^{I^v \cup I^h}. \end{aligned}$$

Definition 2.17. Given a stratum C of a model \mathcal{X} , which corresponds to the stratum τ_C in $\Sigma_{(\mathcal{X}, \mathcal{H})}$, let $\text{Star}(\tau_C)$ denote the *open star* of τ_C inside $\Sigma_{(\mathcal{X}, \mathcal{H})}$.

Example 2.9 admits another formulation in the context of snc log pairs.

Proposition 2.18. [MN15, Proposition 2.4.6] Let $(\mathcal{Y}, \mathcal{D})$ be a snc log model for $U = Y \setminus D$ where $\mathcal{Y}_K \cong Y$ and $\mathcal{D}_K \cong D$. Take a point $x = (x_1, \dots, x_n) \in \Sigma_{(\mathcal{Y}, \mathcal{D})} \subseteq \mathbb{R}^{I^v \cup I^h}$ and let $I = \{i \in I^v \cup I^h : x_i \neq 0\}$. Let ξ_x be a generic point of the intersection $\bigcap_{i \in I} \mathcal{D}_i$. Then there exists a unique real valuation

$$(1) \quad v_x : \mathcal{O}_{\mathcal{Y}, \xi_x} \setminus \{0\} \longrightarrow \mathbb{R}$$

such that $v_x(f_i) = x_i$ for all $i \in I^v \cup I^h$, where f_i is a local equation for \mathcal{D}_i in $\mathcal{O}_{\mathcal{Y}, \xi_x}$. The valuation v_x extends the valuation on K .

The valuation constructed in Proposition 2.18 defines a birational point of Y and thus also defines a birational point of U .

Proposition 2.19. Let $(\mathcal{Y}, \mathcal{D})$ be an snc log model of U . Then there is a canonical embedding $\iota_{(\mathcal{Y}, \mathcal{D})} : \Sigma_{(\mathcal{Y}, \mathcal{D})} \hookrightarrow U^{\text{an}}$ with a continuous retraction map $\rho_{(\mathcal{Y}, \mathcal{D})} : U^{\text{an}} \rightarrow \Sigma_{(\mathcal{Y}, \mathcal{D})}$.

Note that Proposition 2.19 also holds in the case $D = 0$ and recovers the original theory of retractions. Again, in this case we will write $\iota_{\mathcal{Y}}$ and $\rho_{\mathcal{Y}}$ respectively.

Proof. The embedding is given by Proposition 2.18. The map $\rho_{(\mathcal{Y}, \mathcal{D})} : U^{\text{an}} \rightarrow \Sigma_{(\mathcal{Y}, \mathcal{D})}$ is defined as follows. We first formally complete the R -scheme \mathcal{Y} along its special fibre to get a formal scheme $\widehat{\mathcal{Y}}$. Then by §2.3, and since \mathcal{Y} is proper, we have the reduction map

$$\text{red}_{\mathcal{Y}} : Y^{\text{an}} \longrightarrow \mathcal{Y}_k.$$

We can restrict $\text{red}_{\mathcal{Y}}$ to $U^{\text{an}} \subset Y^{\text{an}}$ to get a map $\text{red} : U^{\text{an}} \longrightarrow \mathcal{Y}_k$. Given a point $x \in U^{\text{an}}$, let E_1, \dots, E_n be the irreducible components of \mathcal{Y}_k and E_{n+1}, \dots, E_m the irreducible components of \mathcal{D} passing through the point $\text{red}(x)$. Let ξ_x be the generic point of the connected component in the intersection $\cap_i E_i$ containing $\text{red}(x)$. Then $\rho_{(\mathcal{Y}, \mathcal{D})}(x)$ is the point in $C_{(\mathcal{Y}, \mathcal{D})}$ given by $(v_x(f_i)) \in \mathbb{R}^{I^v \cup I^h}$ where f_i is a local equation for E_i in $\mathcal{O}_{\mathcal{Y}, \text{red}(x)}$. Since $v_x(t) = 1 = \sum_{i \in I^v} N_i v_x(f_i)$, the point $\rho_{\mathcal{Y}, \mathcal{D}}(x)$ lies in $\Sigma_{(\mathcal{Y}, \mathcal{D})}$. By construction, the retraction map is continuous and the right inverse to the embedding $\iota_{(\mathcal{Y}, \mathcal{D})}$. \square

Example 2.20. Let \mathcal{X} be a snc model of X and Y a stratum of \mathcal{X}_k . Then the (analytic) generic fibre in the sense of Berkovich, which we denote $]Y[$, of the formal completion $\widehat{\mathcal{X}}/Y$ along Y is described as

$$]Y[= \{x \in X^{\text{an}} : \text{red}_{\mathcal{X}}(x) \in Y\}.$$

Let $\tau_Y \subset \Sigma_{\mathcal{X}}$ denote the stratum in the skeleton $\Sigma_{\mathcal{X}}$ corresponding to Y . Then

$$\rho_{\mathcal{X}}^{-1}(\text{Star}(\tau_Y)) =]Y[\subseteq X^{\text{an}}$$

and there is a retraction map

$$\rho_Y :]Y[\rightarrow \text{Star}(\tau_Y)$$

constructed as in the proof of Proposition 2.19. The retraction ρ_Y coincides with the map $\rho_{\mathcal{X}}$; in particular, the retraction over $\text{Star}(\tau_Y)$ only depends on the formal scheme $\widehat{\mathcal{X}}/Y$.

Remark 2.21. The choice of snc log model is not unique in the following sense.

Let $C'_{(\mathcal{X}, \mathcal{H})}$ be a finite rational polyhedral subdivision of $C_{(\mathcal{X}, \mathcal{H})}$ into simplicial cones whose integer points can be generated by a subset of a basis of $\mathbb{Z}^{I^v \cup I^h}$. Then by [KKMSD73, Chapter II + Chapter IV], there is an snc pair $(\mathcal{X}', \mathcal{H}')$ and a proper morphism $\mathcal{X}' \rightarrow \mathcal{X}$ which induces an isomorphism $(\mathcal{X}' \setminus \mathcal{H}')_{\eta} \rightarrow (\mathcal{X} \setminus \mathcal{H})_{\eta}$ such that

$$C_{(\mathcal{X}', \mathcal{H}')} \cong C'_{(\mathcal{X}, \mathcal{H})}, S_{(\mathcal{X}', \mathcal{H}')} \cong S'_{(\mathcal{X}, \mathcal{H})}$$

where $S'_{(\mathcal{X}, \mathcal{H})}$ is the subdivision of $S_{(\mathcal{X}, \mathcal{H})}$ induced by the subdivision $C'_{(\mathcal{X}, \mathcal{H})}$.

Conversely, suppose we blow up an snc pair $(\mathcal{X}, \mathcal{H})$ at a connected component of an intersection of irreducible components of \mathcal{X}_k . Denote the new pair by $(\mathcal{X}', \mathcal{H}')$ with \mathcal{H}' the strict transform

of \mathcal{H} . Then $C_{(\mathcal{X}', \mathcal{H}')} and $S_{(\mathcal{X}', \mathcal{H}'})$ are finite rational polyhedral subdivisions of $C_{(\mathcal{X}, \mathcal{H})}$ and $S_{(\mathcal{X}, \mathcal{H})}$ respectively.$

2.5. Essential skeleton of a pair. In the introduction, we briefly discussed the essential skeleton of a K -variety with respect to a global non-vanishing volume form ω . Here, we describe a relative notion following [BM19, §5.1] in a more general setting.

Definition 2.22. An *snc pair over K* is a pair (X, Δ_X) where X is a proper and regular K -scheme and $\Delta_X = \sum_i \Delta_{X,i}$ an effective \mathbb{Q} -Cartier divisor such that $\Delta_{X,i}$ is a prime divisor and the support of Δ_X is snc.

Let $\omega \in K_X^{\otimes m}$ be a regular m -pluricanonical form on X with poles of order at most m along $\Delta_{X,i}$ for each i . Thus ω is a section of $\mathcal{O}_X(m(K_X + \Delta_X))$ and these forms are called Δ_X -logarithmic m -pluricanonical forms.

If \mathcal{X} is a reduced snc model of X , the form ω is a rational section of $\omega_{\mathcal{X}/R}^{\otimes m} \otimes \mathcal{O}_{\mathcal{X}}(m\overline{\Delta}_X)$ where $\overline{\Delta}_X = \sum_i \overline{\Delta}_{X,i}$. In particular, it defines a divisor $\text{div}_{\mathcal{X}}(\omega)$ on \mathcal{X} .

Alternatively, viewing ω as a rational section of $\omega_{X/K}^{\otimes m}$, we can then associate a weight function $\text{wt}_{\omega} : X^{\text{an}} \rightarrow \mathbb{R}$ as in [MN15, BM19]. The weight function on divisorial points behaves as described in the introduction. Given an snc pair $(\mathcal{X}, \mathcal{H})$ and x a divisorial point of $\Sigma_{(\mathcal{X}, \mathcal{H})}$ (under the embedding $\iota_{(\mathcal{X}, \mathcal{H})}$) with corresponding valuation v_x , we have $\text{wt}_{\omega}(x) = v_x(f) + m$ where $f \in K(X)^{\times}$ and $\text{div}_{\mathcal{X}}(\omega) = \text{div}(f)$ locally around $\text{red}_{\mathcal{X}}(x)$.

Definition 2.23. [MN15, §7] The *Kontsevich-Soibelman skeleton* $\text{Sk}(X, \Delta_X, \omega)$ is the closure in $\text{Bir}(X)$ of the set $\text{Div}(X)$ of divisorial points where the weight function reaches its minimal value $\text{wt}_{\omega}(X, \Delta_X) = \inf\{\text{wt}_{\omega}(x) : x \in \text{Div}(X)\} \in \mathbb{R} \cup \{-\infty\}$.

Lemma 2.24. Let ω be a Δ_X -logarithmic pluricanonical form. Then for any snc pair $(\mathcal{X}, \mathcal{H})$ of X , we have the inclusion $\text{Sk}(X, \Delta_X, \omega) \subseteq \Sigma_{(\mathcal{X}, \mathcal{H})}$.

Proof. This is a simpler case of Proposition 4.1.6 in [BM19]. \square

Example 2.25. Let $X = \mathbb{P}_K^1$ and $\Delta_X = 0 + \infty$. Let x and y be affine coordinates on \mathbb{P}^1 and consider the form $\omega = \frac{dx}{x} = -\frac{dy}{y}$. Let $\mathcal{X} = \mathbb{P}_R^1$ and $\overline{\Delta}_X = 0 + \infty$. Then $\Sigma_{\mathcal{X}, \overline{\Delta}_X}$ is isomorphic to \mathbb{R} . Since $\text{div}_{\mathcal{X}}(\omega) = 0$, the weight associated to ω is minimal at every point of $\Sigma_{(\mathcal{X}, \overline{\Delta}_X)}$ and thus $\text{Sk}(X, \Delta_X, \omega) = \Sigma_{(\mathcal{X}, \overline{\Delta}_X)} \cong \mathbb{R}$. It follows from elementary results in [MN15] that the essential skeleton $\text{Sk}(\mathbb{P}^1)$ is empty since $K_X \cong \mathcal{O}(-2)$. The contribution of the horizontal divisor Δ_X gives a non-empty (and non-compact) skeleton of the pair (X, Δ_X) .

Definition 2.26. (Essential skeleton) The *essential skeleton* $\text{Sk}(X, \Delta_X)$ is the union of all Kontsevich-Soibelman skeleta $\text{Sk}(X, \Delta_X, \omega)$ where ω ranges over all regular Δ_X -logarithmic pluricanonical forms.

Proposition 2.27. Let $(\mathcal{X}, \mathcal{D})$ be a reduced snc pair and (X, Δ_X) an snc pair over K such that $\mathcal{X} \times_R K = X$ and $\mathcal{D} = \overline{\Delta}_X$ where $\overline{\Delta}_X$ is the scheme-theoretic closure of Δ_X in \mathcal{X} . Suppose that $K_X + \Delta_X$ and $K_{\mathcal{X}} + \overline{\Delta}_X$ are semiample. Then, under the embedding $\iota_{(\mathcal{X}, \mathcal{D})}$ from Proposition 2.19, $\Sigma_{(\mathcal{X}, \mathcal{D})} = \text{Sk}(X, \Delta_X) \subseteq X^{\text{an}}$.

Proof. This is again a simpler case of Proposition 5.1.7 in [BM19]. \square

Remark 2.28. If $(\mathcal{X}, \mathcal{D})$ is a reduced snc pair as in Proposition 2.27 and $K_X + \Delta_X \sim 0$, then $\mathcal{L} := K_{\mathcal{X}} + \overline{\Delta}_X$ being semiample is equivalent to $K_{\mathcal{X}} + \overline{\Delta}_X \sim 0$. Indeed, since $\mathcal{L}(X) = K^{\times}$, we have that \mathcal{L} must be a multiple of \mathcal{X}_k and hence principal.

2.6. Integral affine structures. We recall the basic notions concerning integral affine structures.

Definition 2.29. An *integral affine function* on an open subset of \mathbb{R}^n is a continuous real-valued function locally of the form $f(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n + b$ with $a_i \in \mathbb{Z}$ and $b \in \mathbb{R}$. We denote the sheaf of integral affine functions on \mathbb{R}^n by $\text{Aff}_{\mathbb{R}^n}$.

Definition 2.30. Let B be a topological manifold. An n -dimensional *integral affine manifold* is a ringed space (B, Aff_B) which is locally isomorphic to $(\mathbb{R}^n, \text{Aff}_{\mathbb{R}^n})$.

We say B is an *integral affine manifold with singularities* if there exists a subset $\Gamma \subseteq B$ which is locally a finite union of closed subspaces of codimension greater than or equal to 2 and $B \setminus \Gamma$ is an integral affine manifold.

Example 2.31. For an snc pair $(\mathcal{X}, \mathcal{H})$, the embeddings of $C_{(\mathcal{X}, \mathcal{H})}$ and $\Sigma_{(\mathcal{X}, \mathcal{H})}$ into $\mathbb{R}^{I^\vee \cup I^h}$ naturally induce piecewise integral affine structures on $C_{(\mathcal{X}, \mathcal{H})}$ and $\Sigma_{(\mathcal{X}, \mathcal{H})}$ respectively, via the lattice embedding $\mathbb{Z}^{I^\vee \cup I^h} \hookrightarrow \mathbb{R}^{I^\vee \cup I^h}$.

If B is an integral affine manifold, then is it also smooth since the transition functions between local models of the ringed space are smooth. Thus given an integral affine manifold B , there is a subsheaf $\Lambda \subset \Lambda_{\mathbb{R}} = TB$ of the tangent sheaf consisting of integral tangent vectors. Similarly, there is a sheaf of integral cotangent vectors $\check{\Lambda} := \mathcal{H}om(\Lambda, \mathbb{Z})$ on B . By definition, an integral affine structure Aff_B fits into the short exact sequence

$$0 \rightarrow \mathbb{R} \rightarrow \text{Aff}_B \rightarrow \check{\Lambda} \rightarrow 0.$$

Remark 2.32. Given an integral affine structure on B , there is an associated $\text{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ -torsor on B whose fibre over a point x consists of all possible integral affine coordinate systems at x .

2.7. Affinoid torus fibrations. Let T be a split multiplicative n -dimensional torus over K with character lattice M and cocharacter lattice $N = M^\vee$.

Definition 2.33. Define the *tropicalisation map* of T as

$$\begin{aligned} \text{trop} : T^{\text{an}} &\longrightarrow N_{\mathbb{R}} \\ |\cdot| &\longmapsto (m \mapsto -\log |m|). \end{aligned}$$

This map is continuous and its fibres are (not necessarily strict) K -affinoid tori.

Example 2.34. Let \mathcal{X} be a regular toric model of T *i.e.* a regular toric scheme over R such that $\mathcal{X}_K = T$. By [Gub13, §7], such a model is described by a regular fan $\Sigma \subset N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ whose cones intersect $N_{\mathbb{R}} \times \{0\}$ only at the origin of $N_{\mathbb{R}}$. Then by the proof of Proposition 2.19, we have a retraction map

$$\rho_{\mathcal{X}} :]\mathcal{X}_k[\longrightarrow \Sigma_{\mathcal{X}}$$

and a canonical embedding $\Sigma_{\mathcal{X}} \hookrightarrow]\mathcal{X}_k[$. Let $\Sigma_1 = \Sigma \cap (N_{\mathbb{R}} \times \{1\})$. Then there is a natural identification of Σ_1 with $\Sigma_{\mathcal{X}}$ given by sending a vertex of Σ_1 to the primitive generator of the corresponding ray of $\Sigma_{\mathcal{X}}$ and extending linearly [Gub13, §7.9]. Furthermore, we have $] \mathcal{X}_k [= \text{trop}^{-1}(|\Sigma_1|)$ and $\rho_{\mathcal{X}}$ is the restriction of trop over $] \mathcal{X}_k [$ [MPS23, Proposition 1.5.2].

When X_{Σ} is an n -dimensional T -toric variety given by a fan Δ , Kajiwara [Kaj08] and Payne [Pay09] independently constructed a tropicalisation map

$$\text{trop}_{\Sigma} : X^{\text{an}} \longrightarrow N_{\mathbb{R}}(\Delta)$$

where $N_{\mathbb{R}}(\Delta)$ is a partial compactification of $N_{\mathbb{R}} \cong \mathbb{R}^n$ which is uniquely determined by Δ . The space $N_{\mathbb{R}}(\Delta)$ has the structure of an extended cone complex. Restricting the map trop_{Δ} to the multiplicative torus, we recover the usual tropicalisation map, but the base $N_{\mathbb{R}}$ is endowed with the fan structure arising from Δ . The base also has an integral affine structure arising from the natural lattice embedding $\mathbb{Z}^n \subseteq \mathbb{R}^n$.

Definition 2.35. Let X be an algebraic variety over K and B a topological space. We say that a continuous map $\rho : X^{\text{an}} \rightarrow B$ is an *affinoid torus fibration* at $b \in B$ if there exists an open neighbourhood U of b such that the restriction of $\rho^{-1}(U)$ fits into the commutative diagram

$$\begin{array}{ccc} \rho^{-1}(U) & \xrightarrow{\sim} & \text{trop}^{-1}(V) \\ \downarrow \rho & & \downarrow \text{trop} \\ U & \xrightarrow{\sim} & V \end{array}$$

where the top horizontal arrow is an isomorphism of Berkovich analytic spaces, the bottom horizontal arrow is a homeomorphism and V is an open subset of $N_{\mathbb{R}} \cong \mathbb{R}^n$.

Remark 2.36. Affinoid torus fibrations are the analogue of smooth real torus fibrations over \mathbb{C} . As suggested in [Pay09, Remark 3.3], the tropicalisation map is a non-archimedean analogue of the moment map.

Given a continuous map $\rho : X^{\text{an}} \rightarrow B$, let B^{sm} denote the locus of points where ρ is an affinoid torus fibration. Then there is an induced integral affine structure on B^{sm} given by the pullback of the integral affine structure on \mathbb{R}^n via the charts given in Definition 2.35. There will be a useful alternate characterisation of the integral affine structure on B^{sm} , due to Kontsevich-Soibelman [KS06]. Let U be a connected open subset of B^{sm} and h an invertible function on $\rho^{-1}(U)$. Then by the maximum modulus principle, the modulus $|h|$ is constant on the fibres and descends to a continuous function on B^{sm} . After sheafifying, we then have the sheaf

$$\text{Aff}_{B^{\text{sm}}, \mathbb{Z}}(U) = \{-\log |h| : h \in \mathcal{O}_{X^{\text{an}}}^{\times}(\rho^{-1}(U))\}.$$

In §5, we will review this construction in more detail.

Definition 2.37. Given an snc log model $(\mathcal{X}, \mathcal{H})$ of X , we say that $\rho_{(\mathcal{X}, \mathcal{H})}$ is a *non-archimedean SYZ fibration* associated to $(\mathcal{X}, \mathcal{H})$ if the discriminant locus $\Gamma := \Sigma_{(\mathcal{X}, \mathcal{H})} \setminus \Sigma_{(\mathcal{X}, \mathcal{H})}^{\text{sm}}$ has codimension greater than or equal to 2 in $\Sigma_{(\mathcal{X}, \mathcal{H})}$.

In §3.7, we will use Examples 2.20 and 2.34 in the construction of a non-archimedean SYZ fibration for log Calabi-Yau surfaces. To achieve this, we will show that our fibration is in fact an affinoid torus fibration over 1-dimensional strata in the skeleton by constructing a toric model \mathcal{X} of \mathbb{G}_m^2 and a toric divisor $C \subset \mathcal{X}$ which, when we take the formal completion $\widehat{\mathcal{X}}_{/C}$, will be isomorphic to the formal completion of the original snc log model completed along the corresponding strata in the special fibre.

3. LOG CALABI-YAU SURFACES

In this chapter, we will introduce log Calabi-Yau surfaces and describe their moduli. Log Calabi-Yau surfaces are non-compact analogues of Calabi-Yau surfaces and have been extensively studied in mirror symmetry [GHK15a, HK22, LZ23]. The main result of this chapter is the construction of

a non-archimedean SYZ fibration associated to a log Calabi-Yau surface which satisfies a natural technical assumption (3.22).

In §3.1, we review the theory of log Calabi-Yau surfaces following [Fri16, GHK15b] and give a simple classification of them (3.10) as certain blowups of toric surfaces. In addition, we collect and prove several technical results that will be needed for the period comparison (6.10). In §3.2, we define the algebraic period map of a log Calabi-Yau surface (3.21) and explain how the algebraic period recovers the familiar notion of a period integral of a 2-form over a singular cycle. In §3.3, we will construct an snc log model of a log Calabi-Yau surface which will be central to this thesis. We explain the operations needed to construct these models and describe both the homeomorphism type and simplicial structure of the skeleton of such models (3.32). We also show that the skeleton of these snc log models coincides with the essential skeleton of the original Looijenga pair. Finally, using the constructions from the section, we describe an explicit model (3.34) which will be used for the construction of a K -affine structure. The explicit construction of the model is a crucial ingredient to prove that the domains of the algebraic and non-archimedean period map agree. In §3.7, we prove that the retraction map associated to the model above is indeed a non-archimedean SYZ fibration in the sense of Definition 2.37. This requires proving a technical lemma (3.37) which roughly tells us that the infinitesimal neighbourhood of a 1-dimensional stratum in the model looks like a toric variety. The proof of the lemma gives a construction of an explicit local toric model of a neighbourhood of a 1-dimensional stratum in the model; this will be very useful in future computations. With the construction of the non-archimedean SYZ fibration, we can compute the induced integral affine structure on the smooth locus of the fibration.

3.1. The geometry of log Calabi-Yau surfaces. We recall the following definition given in the introduction:

Definition 3.1. A *Looijenga pair* over a field F of characteristic 0 is a couple (Y, D) where Y is a smooth projective surface over F and $D \in |-K_Y|$ is a reduced nodal curve with at least one singular point.

We will write $D = D_1 + \dots + D_n$ where D_1, \dots, D_n are the irreducible components of D . By our definition, D is either an irreducible nodal curve or a cycle of rational curves, meaning that D_i only intersects $D_{i\pm 1}$ transversally where i is taken modulo n . Let $\ell(D)$ denote the number of irreducible components of D .

Let $U = Y \setminus D$. There is a global non-vanishing 2-form $\omega \in H^0(Y, K_Y)$ on Y which has simple poles along the divisor D ; this form is unique up to scaling. The K -variety U is a *log Calabi-Yau surface* with maximal boundary. For brevity, we will just refer to $U = Y \setminus D$ as a log Calabi-Yau surface.

Example 3.2. Let Y be a smooth toric surface over F and D the toric boundary of Y . Then (Y, D) is a Looijenga pair, which we will refer to as a *toric Looijenga pair*. We can blow up any smooth point of D to produce a new surface \tilde{Y} and take the strict transform \tilde{D} of D . The new pair (\tilde{Y}, \tilde{D}) is also a Looijenga pair.

Remark 3.3. The algebraic closure of K is given by the field of Puiseux series $\bar{K} = \mathbb{C}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{C}((t^{\frac{1}{n}}))$. Throughout this paper, we will blow up points of D which may be closed *i.e.* K' -rational points where K' is a finite extension of K . For this reason, we will make a finite field extension of K of the form $K_n = \mathbb{C}((t^{\frac{1}{n}}))$ such that the points we are blowing up become rational. Since K_n is isomorphic to K as discrete valuation fields, given a Looijenga pair (Y, D) we will fix such a finite extension of K and abuse notation to denote these finite extensions also by K and its

ring of integers by R . By the birational geometry of surfaces over an algebraically closed field of characteristic 0, we can further assume that Y is rational.

Definition 3.4. A *Looijenga pair over R* is a pair $(\mathcal{Y}, \mathcal{D})$ where \mathcal{Y} is a smooth projective rational surface over R and $\mathcal{D} \in |-K_{\mathcal{Y}/R}|$ is nodal. We say that $(\mathcal{Y}, \mathcal{D})$ has *good reduction* if both the generic and special fibre are Looijenga pairs over K and k respectively.

Convention 3.5. A Looijenga pair over K will be denoted by (Y, D) and a Looijenga pair over R will be denoted by $(\mathcal{Y}, \mathcal{D})$.

In fact, Example 3.2 entirely captures the geometry of Looijenga pairs. To see this, we make the following definition:

Definition 3.6. For a Looijenga pair (Y, D) over K , a blowup at a node of D is a *toric blowup* of Y . A *non-toric blowup* is a blowup of a smooth point of D .

Definition 3.7. Let (Y, D) be a Looijenga pair over K . An irreducible curve E on Y is an *interior exceptional curve* if $E \cong \mathbb{P}^1$, $E^2 = -1$ and $E \neq D_i$ for any i .

Definition 3.8. Let (Y, D) be a Looijenga pair over K . A *toric model* for (Y, D) consists of a toric Looijenga pair (\bar{Y}, \bar{D}) and a birational morphism $\pi : Y \rightarrow \bar{Y}$ such that $D \rightarrow \bar{D}$ is an isomorphism.

In the above definition of toric model, we account for the case the smooth points that are blown up on D are infinitely near *i.e.* having blown up a point $p \in \bar{D}_i$, we then blow up the intersection of the new exceptional curve E with the strict transform of \bar{D}_i and so on. This leads to the following definition of generalised exceptional curve:

Definition 3.9. A *generalised exceptional curve* on Y is a divisor of the form E or $C_1 + \dots + C_k + E$, where $k \geq 1$, $C_i \cong \mathbb{P}^1$ is a smooth curve of self-intersection -2 disjoint from D for $i \leq k - 1$, E is an interior exceptional curve with the relations $C_i \cdot C_j = 1$ if $j = i \pm 1$ and 0 otherwise, $E \cdot C_k = 1$ and $E \cdot C_i = 0$ for $i \neq k$.

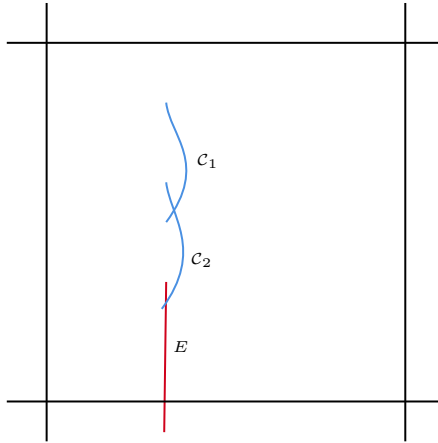
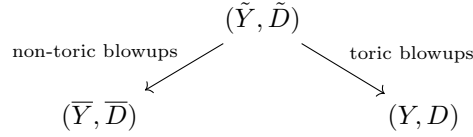


FIGURE 2. Example of a generalised exceptional curve

Proposition 3.10. [GHK15a, Proposition 1.3] Let (Y, D) be a Looijenga pair over K . After a finite extension of K , there exists a Looijenga pair (\tilde{Y}, \tilde{D}) , a toric blowup of (Y, D) , such that the pair (\tilde{Y}, \tilde{D}) admits a toric model (\bar{Y}, \bar{D}) *i.e.* we have the following diagram



In light of Remark 3.3, the above proposition also holds over a finite extension of $K = \mathbb{C}((t))$. Indeed, a minimal smooth projective rational surface over an algebraically closed field of characteristic 0 is either \mathbb{P}^2 or a ruled surface. When we drop the assumption that the field is algebraically closed, but assume it is still perfect, every minimal rational surface is either a del Pezzo surface with Picard rank 1 or a conic bundle with Picard rank 2 by [Isk79] [citation?](#) and [MT86, Theorem 3.3.1]. Given a Looijenga pair (Y, D) , we can make a large enough field extension $K_n = \mathbb{C}((t^{\frac{1}{n}}))$ such that the minimal model of Y over K_n is either \mathbb{P}^2 or a ruled surface. Then the proof in [GHK15a] proceeds as usual.

The constructions that appear in this paper are independent of the choice of compactification of U and thus will not be affected by toric blowups. For this reason, we will always assume the necessary toric blowups have been performed so that any Looijenga pair (Y, D) has a toric model (\bar{Y}, \bar{D}) and a birational morphism $\pi : (Y, D) \rightarrow (\bar{Y}, \bar{D})$ which is a composition of finitely many non-toric blowups. **Should this be made more prominent? not sure. OR just say we will often pass to a blow up with a toric model and just make it clear whenever we use it.**

Definition 3.11. Let (Y, D) be a Looijenga pair over K .

- (1) A curve $C \subset Y$ is *interior* if no irreducible component of C is contained in D ;
- (2) An *internal* (-2) -curve is a smooth rational curve with self-intersection -2 which is disjoint from D ;
- (3) A pair (Y, D) is *generic* if it has no internal (-2) -curves.

By Proposition 4.1 in [GHK15b], any Looijenga pair is deformation equivalent to a generic pair. By adjunction, any smooth rational curve with negative self-intersection is either a (-1) -curve meeting D transversally at a single point or a (-2) -curve.

Example 3.12. (i) Let (Y, D) be a Looijenga pair given by a finite number of non-toric blow ups of (\bar{Y}, \bar{D}) . Then (Y, D) is generic if and only if the centres of the non-toric blowups on (\bar{Y}, \bar{D}) are distinct.

- (ii) Given a generalised exceptional curve $C_1 + \dots + C_{k-1} + E$, the curves C_i are internal (-2) -curves as depicted in Figure 2.

Definition 3.13. Let (Y, D) be a generic Looijenga pair with toric model (\bar{Y}, \bar{D}) .

- (1) An *exceptional configuration* for (Y, D) is a collection $\{E_{ij}\} \subset \text{Pic}(Y)$ of classes of exceptional divisors arising from non-toric blowups of the toric model. Here, E_{ij} are the exceptional divisors of the toric model meeting the divisor D_i . We shall refer to them as *non-toric exceptional curves*.
- (2) Let (Y, D) and (Y', D') be two generic Looijenga pairs with the same toric model (\bar{Y}, \bar{D}) with respective exceptional configurations $\{E_{ij}\}$ and $\{E'_{ij}\}$. We say that (Y, D) and (Y', D')

have the same *combinatorial type* if for each toric divisor $\overline{D}_i \subset \overline{D}$, the number of exceptional divisors of the toric model for Y and Y' with centre on \overline{D}_i is the same.

Example 3.14. Let (Y, D) be a toric Looijenga pair over k and consider the base change of the pair to R which we denote $(\mathcal{Y}, \mathcal{D})$. Let $\mu_i \in \mathcal{D}_i \times_k K$ be a smooth K -rational point and denote by $(\mathcal{Y}', \mathcal{D}')$ the Looijenga pair over R given by the blowup at the closure of μ_i in \mathcal{Y} . There are two cases to consider:

- (i) the specialisation of μ_i is a nodal point of the divisor D : the special fibre will be another toric Looijenga pair given by a toric blowup of (Y, D) ;
- (ii) the specialisation of μ_i is a smooth point of $D_i \subset D$: the special fibre will have the same combinatorial type as the generic fibre. This case will be the most important for our purposes.

Remark 3.15. Unfortunately, since the special fibre in (i) is a toric Looijenga pair, the methods of this thesis will not be able to deal with this case. This explains the assumption we make in (3.22).

Definition 3.16. Let (Y, D) be a Looijenga pair with $\ell(D) \geq 3$. An *orientation* of D is an orientation of the dual graph of D .

An orientation of D determines and is determined by a labelling of the components up to cyclic permutation. In particular, the labelling above of the components of $D = \sum_i D_i$ determines an orientation.

Remark 3.17. There is another interpretation of the orientation that we will use later. Let $\nu : \tilde{D} \rightarrow D$ be the normalisation of D . Then given the singular points of D , which we denote by D^{sing} , we can consider the possible labellings of the set $\nu^{-1}(D^{\text{sing}})$ by 0 and ∞ subject to the following conditions:

- (i) if $q_1, q_2 \in \nu^{-1}(D^{\text{sing}})$ and $\nu(q_1) = \nu(q_2)$ then exactly one of the q_i is labelled by 0 and the other q_j is labelled by ∞ ;
- (ii) if $\tilde{D}_i = \nu^{-1}(D_i)$ for some i and $q_1 \neq q_2 \in \tilde{D}_i$ with $\nu(q_1), \nu(q_2) \in D^{\text{sing}}$, then exactly one of the q_i is labelled by 0 and the other q_j is labelled by ∞ .

Thus fixing a singular point and a labelling of the normalisation of this point determines an orientation of D .

The following result is well-known but we reproduce it here as we will use it later in computations. Let D_i° denote the smooth locus of $D_i \subset D$.

Lemma 3.18. [Fri16, Lemma 1.6] A choice of orientation of D fixes a canonical isomorphism $\psi : \text{Pic}^0(D) \cong \mathbb{G}_m$. Via this isomorphism, if $p, q \in D_i^\circ$ and we choose an affine coordinate on D_i such that p corresponds to 1 and q corresponds to λ , then

$$\psi(\mathcal{O}_D(p - q)) = \lambda^{-1}.$$

Proof. Let $\nu : \tilde{D} \rightarrow D$ denote the normalisation of D and \tilde{D}_i be the component of \tilde{D} which is mapped to D_i . We have the short exact sequence

$$1 \rightarrow \mathcal{O}_D^\times \rightarrow \nu_* \mathcal{O}_{\tilde{D}}^\times \rightarrow \frac{\nu_* \mathcal{O}_{\tilde{D}}^\times}{\mathcal{O}_D^\times} \rightarrow 1.$$

By checking in a neighbourhood of a singular point and away from it, we can identify $\frac{\nu_* \mathcal{O}_{\tilde{D}}^\times}{\mathcal{O}_D^\times}$ with \mathbb{G}_m^ℓ where $\ell = \ell(D)$. Taking long exact sequences in cohomology, we have

$$1 \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m^\ell \rightarrow \mathbb{G}_m^\ell \rightarrow H^1(D, \mathcal{O}_D^\times) \rightarrow \mathbb{Z}^\ell \rightarrow 0$$

since $H^1(\tilde{D}, \mathcal{O}_{\tilde{D}}) \cong \bigoplus_i H^1(\tilde{D}_i, \mathcal{O}_{\tilde{D}_i}) \cong \mathbb{Z}^\ell$. The map $H^1(D, \mathcal{O}_D^\times) \rightarrow \mathbb{Z}^\ell$ is given by the multidegree of the line bundle *i.e.* $L \mapsto (L|_{D_1}, \dots, L|_{D_\ell})$. Thus we have the exact sequence

$$1 \rightarrow \mathbb{G}_m \xrightarrow{\Delta} \mathbb{G}_m^\ell \xrightarrow{f} \mathbb{G}_m^\ell \rightarrow \text{Pic}^0(D) \rightarrow 0$$

with $\Delta(x) = (x, \dots, x)$.

We make the map ψ more explicit now. Let $L \in \text{Pic}^0(D)$ and consider $L_i = L|_{\tilde{D}_i}$ with a trivialisation $\sigma_i : L_i \cong \mathcal{O}_{\tilde{D}_i}$. Given the marked points 0_i and ∞_i on \tilde{D}_i , we can build the line bundle L on D by specifying the gluing maps $\mu_i : (L_i)_{0_i} \cong (L_{i-1})_{\infty_{i-1}}$. Using the trivialisations σ_i , we can identify $(L_i)_{0_i}$ and $(L_{i-1})_{\infty_{i-1}}$ with K and thus each μ_i can be identified with an element of \mathbb{G}_m . Changing the isomorphism σ_i to $x_i \cdot \sigma_i$ for $x_i \in \mathbb{G}_m$ replaces μ_i with $x_i^{-1} x_{i-1} \mu_i$ and thus we have $f(x_1, \dots, x_n) = (x_1^{-1} x_r, \dots, x_r^{-1} x_{r-1})$.

Define $p : \mathbb{G}_m^\ell \rightarrow \mathbb{G}_m$ by $(x_1, \dots, x_\ell) \mapsto x_1 \cdot \dots \cdot x_\ell$. Then $\ker(p) = \text{im}(f)$ and p is clearly surjective, thus identifying $\text{Pic}^0(D)$ with \mathbb{G}_m .

Lastly, consider the line bundle $\mathcal{O}_D(p - q)$ and run through the argument above. The nowhere vanishing global section $\sigma_i = \frac{x-1}{x-\lambda}$ trivialises the line bundle $L_i = \mathcal{O}_{\tilde{D}_i}(p_i - q_i)$. Then in the above notation, we have $\mu_i = \lambda^{-1}$ and $\mu_j = 1$ for $j \neq i$. Thus, $\psi(\mathcal{O}_D(p - q)) = \lambda^{-1}$. \square

Lemma 3.19. [Fri16, Lemma 1.9] Let $\mathbf{d} = \sum_{i,j} (p_{ij} - q_{ij})$ be a divisor of multidegree $(0, \dots, 0)$ with $p_{ij}, q_{ij} \in D_i^\circ$. Let z_i be an affine coordinate on D_i for each i . Then \mathbf{d} is a principal divisor if and only if

$$\prod_{i,j} \frac{z_i(p_{ij})}{z_i(q_{ij})} = 1.$$

3.2. Algebraic periods. In the introduction, we introduced the algebraic period map for a Looijenga pair (Y, D) . In this section, we will recall the definition and briefly explain how the map in fact computes period integrals.

Definition 3.20. Given a Looijenga pair (Y, D) over a field F of characteristic 0, we define

$$D^\perp := \{\alpha \in \text{Pic}(Y) \mid \alpha \cdot [D_i] = 0 \text{ for all } i\}.$$

Definition 3.21. Given an orientation of D , define the *algebraic period map* as

$$\begin{aligned} \mathcal{P}_{(Y,D)} : D^\perp &\longrightarrow \text{Pic}^0(D) \cong_\psi \mathbb{G}_m \\ L &\longmapsto \psi(L|_D) \end{aligned}$$

where ψ is the canonical isomorphism from Lemma 3.18.

We briefly justify why $\mathcal{P}_{(Y,D)}$ is the same as the classical notion of period integral. Let (Y, D) be a Looijenga pair over \mathbb{C} with Ω a volume form on $Y \setminus D$ with simple poles along each irreducible component of D . We choose Ω such that

$$\int_\Gamma \Omega = \frac{1}{(2\pi i)^2},$$

where Γ is the homology class of a real 2-dimensional torus in U . Given $\gamma \in D^\perp$, there exists a class $\tilde{\gamma} = \sum_k \pm [C_k]$ representing γ , where C_k is a smooth curve in Y meeting D transversally at the smooth locus of D **apparently this isn't obvious...we had to add a proof of this to our paper so can cite that**. Then a cycle γ' can be constructed, which is homologous to $\tilde{\gamma}$ and is used to define the period map

$$\begin{aligned} \varphi_{(Y,D)} : D^\perp &\longrightarrow \mathbb{G}_m \\ \gamma &\longmapsto \exp\left(\frac{1}{2\pi i} \int_{\gamma'} \Omega\right). \end{aligned}$$

Then Proposition 3.12 in [Fri16] shows that these integrals are equal to the periods as defined in Definition 3.21. By [Car85], the mixed Hodge structure of U is classified by such integrals.

3.3. Construction of the model and skeleton. In §2, we defined an snc log model of a variety and constructed the skeleton associated to it. In this section, given a generic Looijenga pair (Y, D) over K which arises from the generic fibre of a Looijenga pair $(\mathcal{Y}, \mathcal{D})$ over R with good reduction (Definition 3.4), we will describe several birational modifications. We will apply these to $(\mathcal{Y}, \mathcal{D})$ to explicitly construct an snc log model of $U = Y \setminus D$ that will be used in the definition of the non-archimedean period map. The constructions we describe are of a very special type in the sense that the homeomorphism type of the associated skeleton is fixed. We will also give a detailed description of the simplicial structure of the associated skeleton in this section.

3.4. Birational modifications. In light of Example 3.14, we make the following assumption.

Assumption 3.22. Given a generic Looijenga pair (Y, D) over K , we will assume that the pair can be realised as a generic fibre of a Looijenga pair $(\mathcal{Y}, \mathcal{D})$ over R with good reduction where \mathcal{D} is the scheme-theoretic closure of D inside \mathcal{Y} . Furthermore, we will assume the special fibre $(\mathcal{Y}_k, \mathcal{D}_k)$ is a generic Looijenga pair over k with the same combinatorial type as the generic fibre $(\mathcal{Y}_K, \mathcal{D}_K)$.

In concrete terms, the assumption guarantees that the centres of the non-toric blowups on the generic fibre are distinct and specialise to distinct smooth points in the boundary \mathcal{D}_k of the special fibre \mathcal{Y}_k .

Setup 3.23. Let (Y, D) be a generic Looijenga pair over K , with toric model (\bar{Y}, \bar{D}) satisfying Assumption 3.22. By the assumption, there exists a Looijenga pair $(\mathcal{Y}, \mathcal{D})$ over R with good reduction such that $(\mathcal{Y}, \mathcal{D}) \times_R K = (Y, D)$. We further suppose there are k_i non-toric blowups of distinct K -rational points with centre on \bar{D}_i , which gives rise to the pair (Y, D) . Let \mathcal{D}_i denote the scheme-theoretic closure of D_i in \mathcal{Y} .

We now discuss the birational modifications we will be using in this thesis to construct snc log models of $U = Y \setminus D$. The first construction is the following:

Construction 3.24. If the centre of a non-toric blowup is on \bar{D}_i , we blow up the subvariety $\mathcal{D}_i \times_R k$ inside \mathcal{Y} . Denote the blowup by \mathcal{Y}' . More generally, given a Looijenga pair $(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})$ over R , let $\tilde{\mathcal{D}}_i$ denote the scheme-theoretic closure of D_i in $\tilde{\mathcal{Y}}$. We then blow up $\tilde{\mathcal{D}}_i \times_R k$ in $\tilde{\mathcal{Y}}$.

We can alternatively describe this modification as a deformation to the normal cone of the boundary divisor D_i . In Figure 3, we illustrate the modification on the special fibre by applying Construction 3.24.

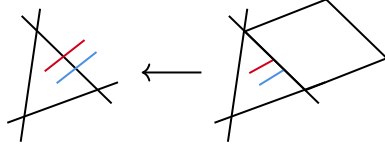


FIGURE 3. Example of Construction 3.24

The next result describes the exceptional divisor in \mathcal{Y}' produced by Construction 3.24. We will deal with the more general case of Construction 3.24 in Lemma 3.31.

Lemma 3.25. Suppose we are in Setup 3.23. Let the self-intersection number of the divisor D_i in Y be n_i . Let \mathcal{Y}' denote the blowup of the subvariety $\mathcal{D}_i \times_R k$ in \mathcal{Y} . Then the new irreducible component Y_1^i produced in the special fibre is isomorphic to $\mathbb{F}_{n_i} = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(n_i))$.

Proof. Let $C \subset \mathcal{Y}_k$ denote the 1-dimensional subvariety given by $\mathcal{D}_i \times_R k$. The curve C is a complete intersection of the special fibre \mathcal{Y}_k and the horizontal divisor \mathcal{D}_i . Thus the conormal bundle $\mathcal{N}_{C/\mathcal{Y}}^\vee$ is equal to $\mathcal{O}_C(-\mathcal{Y}_k) \oplus \mathcal{O}_C(-\mathcal{D}_i)$. Since \mathcal{Y}_k is principal and $\mathcal{D}_i \cdot C = n_i$, this establishes the result. \square

We next review the notion of a type I modification from [Kul77] which we shall refer to just as a *flop*. Further details can be found in [Kul77, §4]. We illustrate the construction in Figure 3.4. Let $\tilde{\mathcal{Y}}$ be an snc model of Y . Suppose we have two irreducible components X, X' in the special fibre of $\tilde{\mathcal{Y}}$ and let $C \subset X \cap X'$ be a double curve. Let E be a curve in X with $E \cong \mathbb{P}^1$ and $E^2 = -1$; further suppose that E does not contain any triple points (meaning a point that is the intersection of three irreducible components), intersects C at a single point P i.e. $(E \cdot C)_X = 1$ and does not intersect any other double curves. Then there exists a birational modification $\tilde{\mathcal{Y}}'$ of $\tilde{\mathcal{Y}}$ such that the exceptional curve E is blown down to a point and the point P on X' is blown up to a curve E' with $E' \cong \mathbb{P}^1$ and $E'^2 = -1$ **is this accurate? the ruling explanation after should suffice just remove**. To see this, we blow up E in $\tilde{\mathcal{Y}}$ to produce an exceptional divisor V isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ whose rulings are given by E and E' and which has multiplicity 1 in the special fibre. We will prove this in Lemma 3.31 in the case of interest to us. The normal bundle of V in $\tilde{\mathcal{Y}}'$ is $\mathcal{O}_V(V)$ and its restriction to the fibre E has degree -1 since $\mathcal{O}_V(V) = \mathcal{O}_V(-E - E')$. Hence we can blow down V to E' along E . Thus we have *flopped* the curve E on X to the curve E' on X' . We will give explicit equations for the contraction of the ruling in §5.5

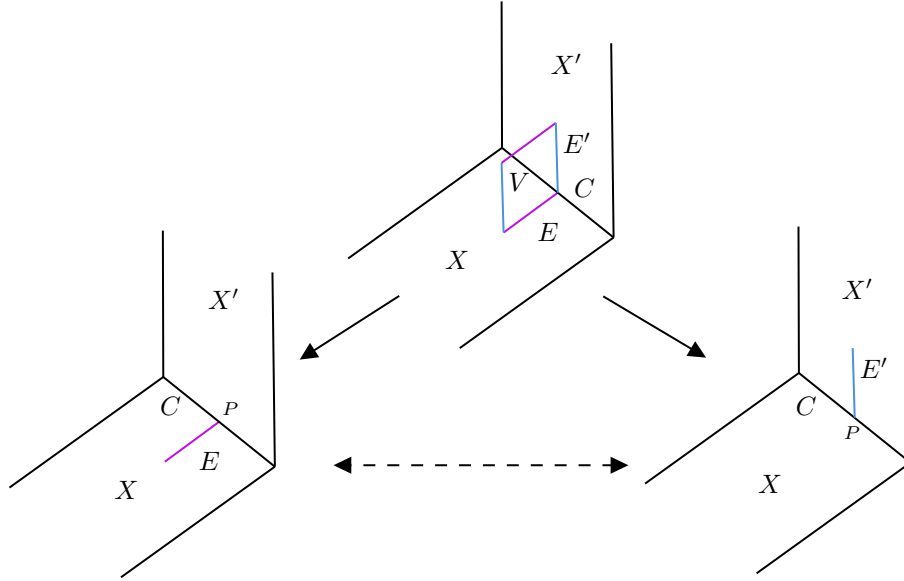
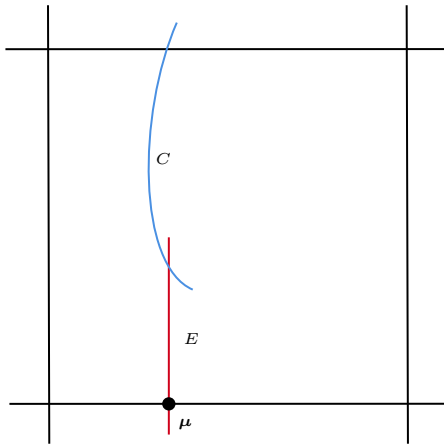


FIGURE 4. Type I modification from [Kul77]

Setup 3.26. Let (Y, D) be a toric Looijenga pair where Y is a ruled surface with D_2 and D_4 fibres of Y under the ruling. Consider the non-toric blowup of Y at a smooth point μ on either D_1 or D_3 and denote the exceptional curve by E . Let C denote the strict transform of the fibre of Y which passes through μ .



The next two constructions make use of the flop we described above.

Construction 3.27. (i) Suppose we have applied Construction 3.24 to the pair $(\mathcal{Y}, \mathcal{D})$. Let E be a non-toric exceptional curve in Y whose centre is on \overline{D}_i . Then E corresponds to an exceptional non-toric curve E_k in \mathcal{Y}_k by assumption. We flop E_k to the newly produced

irreducible component Y' . Denote the new Looijenga pair by $(\mathcal{Y}', \mathcal{D}')$ where \mathcal{D}' is the strict transform of \mathcal{D} . We let \mathcal{D}'_i denote the strict transform of \mathcal{D}_i in \mathcal{Y}' .

- (ii) By Lemma 3.25, the new irreducible component is isomorphic to a Looijenga pair (Y', D') over k whose toric model is a ruled toric surface. Moreover, the centres of the non-toric blowups on Y' are along an irreducible divisor in D' and in particular we are in Setup 3.26. We first apply Construction 3.24 *i.e.* blow up $\mathcal{D}'_i \times_R k$ in \mathcal{Y}' . Then, given a non-toric exceptional curve $E \subset Y'$, let C be the strict transform of the fibre through the centre of E as in Setup 3.26. We flop the curve C to the new irreducible component Y'' . Under this modification, E becomes a fibre of Y'' .

We give an example to illustrate the constructions. **red solid arrows correct in the diagram below**

Example 3.28. We will only draw the special fibre below. In Figure 5, we depict first applying Construction 3.24 and then Construction 3.27 (i) to three non-toric exceptional curves. We then apply Construction 3.27 (ii) to two curves which are strict transforms of the fibres through centres of non-toric blowups.

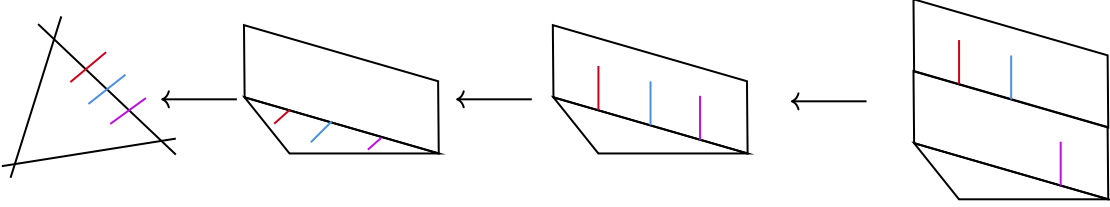


FIGURE 5. Example of Constructions 3.24 and 3.27

dashed arrows

After applying Construction 3.27, we produce a new pair $(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})$. We can apply Construction 3.24 to $\tilde{\mathcal{D}}_i$, the strict transform of \mathcal{D}_i , and proceed to flop the curves C as in Construction 3.27 (ii) to this new irreducible component. We thus consider the following setup:

Setup 3.29. Suppose $\pi : \tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ is a composition of blowups and flops described in Constructions 3.24 and 3.27 with $\tilde{\mathcal{Y}}$ reduced. Then $(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})$ is an snc log model for $U = Y \setminus D$ where $\tilde{\mathcal{D}}$ is the strict transform of \mathcal{D} under π . Let $\tilde{\mathcal{D}}_i$ denote the strict transform of \mathcal{D}_i .

Remark 3.30. The modifications in Constructions 3.24 and 3.27 are of a very special type in the sense that the birational morphism $f : (\tilde{\mathcal{Y}}, \tilde{\mathcal{D}}) \rightarrow (\mathcal{Y}, \mathcal{D})$ is log crepant *i.e.* $f^*(K_{\mathcal{Y}} + \mathcal{D}) = K_{\tilde{\mathcal{Y}}} + \tilde{\mathcal{D}}$. Since $K_{\mathcal{Y}} + \mathcal{D} \sim 0$, we will see in Proposition 3.32 that this fixes the homeomorphism type of the skeleton.

Proposition 3.31. Given a non-toric exceptional curve E in Y , let \mathcal{E} denote the scheme-theoretic closure of E in \mathcal{Y} . Then suppose we are in Setup 3.29 and C is either

- (i) the intersection of the special fibre and $\tilde{\mathcal{D}}_i$ for some i with $k_i > 0$;
- (ii) a non-toric exceptional curve in an irreducible component X of $\tilde{\mathcal{Y}}_k$ given by the intersection of the special fibre and the strict transform of \mathcal{E} .

Then the exceptional divisor of the blowup of C inside $\tilde{\mathcal{Y}}$ is isomorphic to

- (i) $\mathbb{F}_{|C \cdot \tilde{\mathcal{D}}_i|}$
- (ii) $\mathbb{P}^1 \times \mathbb{P}^1$

respectively.

We note that (i) is a more general case of Construction 3.25.

Proof. In case (i), the curve C is a complete intersection of $\tilde{\mathcal{D}}_i$ and $\tilde{\mathcal{Y}}_k$ and thus the conormal bundle is

$$\mathcal{N}_{C/\tilde{\mathcal{Y}}}^\vee = \mathcal{O}_C(-\tilde{\mathcal{D}}_i) \oplus \mathcal{O}_C(-\tilde{\mathcal{Y}}_k) = \mathcal{O}(-C \cdot \tilde{\mathcal{D}}_i) \oplus \mathcal{O}(-C \cdot \tilde{\mathcal{Y}}_k) = \mathcal{O}(-C \cdot \tilde{\mathcal{D}}_i) \oplus \mathcal{O},$$

since $\tilde{\mathcal{Y}}_k$ is a principal divisor.

In case (ii), the non-toric exceptional curve C in X is a complete intersection of X and the horizontal divisor \mathcal{E} . Thus the conormal bundle

$$\mathcal{N}_{C/\tilde{\mathcal{Y}}}^\vee = \mathcal{O}(-(C \cdot X)) \oplus \mathcal{O}(-(C \cdot \mathcal{E})) = \mathcal{O}(-1) \oplus \mathcal{O}(-1),$$

and the exceptional divisor of the blowup of C inside $\tilde{\mathcal{Y}}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. □

3.5. The associated skeleton. We now turn our attention to the skeleton associated to the snc pairs constructed above. It is important to note there are a number of other modifications we could have performed.

If we only apply the modifications described above, then we can describe the skeleton completely.

Proposition 3.32. Suppose we are in Setup 3.29. Then the skeleton $\Sigma_{(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})}$ is homeomorphic to \mathbb{R}^2 .

Proof. We have $K_{\tilde{\mathcal{Y}}} + \tilde{\mathcal{D}} \sim 0$ since the flop we perform in Construction 3.27 has no effect on $K_{\tilde{\mathcal{Y}}}$. If $\pi : \mathcal{Y}' \rightarrow \mathcal{Y}$ corresponds to blowing up $\mathcal{D}_i \times_R k$ in \mathcal{Y} with exceptional divisor X , then we have $K_{\mathcal{Y}'} = \pi^*(K_{\mathcal{Y}}) + X = \mathcal{D}' - X + X = \mathcal{D}'$ where \mathcal{D}' is the strict transform of \mathcal{D} . Repeating this argument for \mathcal{D}'_i gives $K_{\tilde{\mathcal{Y}}} + \tilde{\mathcal{D}} \sim 0$.

Since $K_{\mathcal{Y}} + D \sim 0$ and $K_{\tilde{\mathcal{Y}}} + \tilde{\mathcal{D}} \sim 0$, by Proposition 2.27, the skeleton $\Sigma_{(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})}$ is equal to the essential skeleton $\text{Sk}(Y, D)$. Similarly $\Sigma_{(\mathcal{Y}, \mathcal{D})}$ is equal to $\text{Sk}(Y, D)$ and thus it is enough to describe the homeomorphism type of $\Sigma_{(\mathcal{Y}, \mathcal{D})}$.

We have that $\Sigma_{(\mathcal{Y}, \mathcal{D})}$ is the cone of the dual graph of the boundary D which can be ‘flattened’ to give a homeomorphism to \mathbb{R}^2 . More precisely, for each node $p_{i, i+1} := D_i \cap D_{i+1}$ of D , we have a 2-dimensional cone $\sigma_{i, i+1}$ in $\Sigma_{(\mathcal{Y}, \mathcal{D})}$ generated by v_i and v_{i+1} . We then glue $\sigma_{i, i+1}$ to $\sigma_{i-1, i}$ along the ray $\rho_i := \mathbb{R}_{\geq 0} v_i$ to obtain the piecewise linear manifold $\Sigma_{(\mathcal{Y}, \mathcal{D})}$ which is homeomorphic to \mathbb{R}^2 and a decomposition $\{\sigma_{i, i+1}\} \cup \{\rho_i\} \cup \{0\}$ for i running through the ordered set indexing the irreducible components of the boundary D .

We now seek an explicit description of the polyhedral structure of $\Sigma_{(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})}$. By Remark 2.21, a toric birational morphism between snc log models of $U = Y \setminus D$ i.e. blowing up connected strata of \mathcal{Y}_k corresponds to a finite rational polyhedral subdivision of $C_{(\mathcal{Y}, \mathcal{D})}$. **is this better?**

Performing Construction 3.24 on \mathcal{D}_i adds a new component Y_1^i to the special fibre. By Proposition 3.31, the new component Y_1^i is isomorphic to \mathbb{F}_n for some n . We can repeatedly apply Construction 3.24 and blow up the strict transform of \mathcal{D}_i for any i ; denote by Y_j^i the new component created when applying Construction 3.24 to the strict transform of \mathcal{D}_i for the j^{th} time. Since Constructions 3.27 (i) and (ii) do not change the intersection pattern of the components of the special fibre and the boundary, they also do not change the skeleton. We will still denote the components by Y_j^i

after the application(s) of Constructions 3.27 (i) and (ii). Using the description of $\Sigma_{(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})}$ in Lemma 2.16, the skeleton will thus consist of:

- $m + 1$ vertices in correspondence with the number of irreducible components of the special fibre. Let v_j^i correspond to the irreducible component Y_j^i and v_0 correspond to Y_0 , the strict transform of Y ;
- m finite edges joining the vertices according to the intersection pattern of the special fibre *i.e.* there is an edge between v_j^i and $v_{j'}^{i'}$ if and only if $i = i'$ and $j = j' \pm 1$ and an edge between v_0 and v_1^i for each i ;
- for each vertex v_j^i , there are two rays emanating from v_j^i arising from the intersection of Y_j^i with $\tilde{\mathcal{D}}$;
- n rays emanating from $v_{k_i}^i$ (or v_0 if (Y, D) is toric) corresponding to the intersection of $\tilde{\mathcal{D}}_i$ with $Y_{k_i}^i$ for each i ;
- for each 0-dimensional stratum p of the special fibre $\tilde{\mathcal{Y}}_k$, there is an open 2-cell whose closure in $\Sigma_{(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})}$ includes the rays or edges whose corresponding strata intersect at precisely p .

This completely describes the simplicial structure of $\Sigma_{(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})}$. □

We give a concrete example of the simplicial structure in the following example.

Example 3.33. Let $(\mathcal{Y}, \mathcal{D})$ be the surface over R given by blowing up \mathbb{P}_R^2 at the closure of distinct K -rational points $\lambda, \mu \in D_1^\circ$ whose specialisations to $\mathcal{D}_1 \times_R k$ are smooth distinct k -points. Then $(\mathcal{Y}, \mathcal{D})$ satisfies Assumption 3.22 and the generic and special fibre are non-toric del Pezzo surfaces of degree 7. Let $(Y, D) = (\mathcal{Y}, \mathcal{D}) \times_R K$. In Figure 6, we illustrate a sequence of modifications described above to produce an snc log model $U = Y \setminus D$.

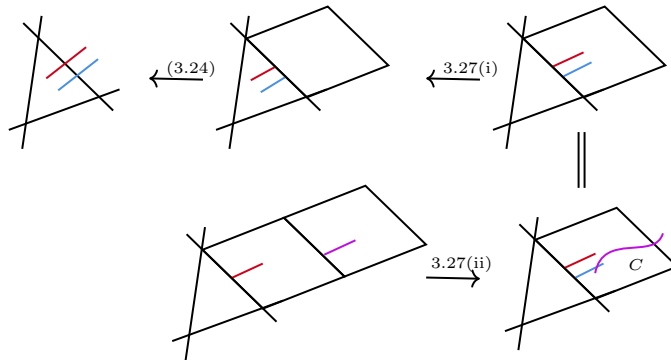
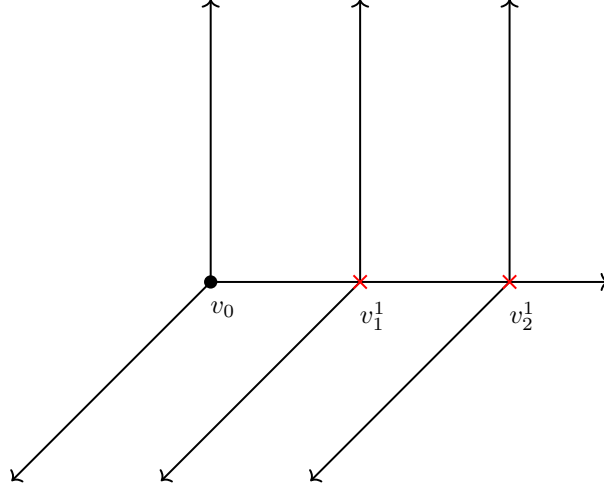


FIGURE 6. Modifications to construct a model for non-toric dP_7

The skeleton of the new pair $(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})$ is illustrated in the figure below; the red crosses depict singularities in the integral affine structure which we discuss in §3.8. You have not talked about what the affine structure is though specifically here



3.6. Construction of the model. One important feature of the above example is that each irreducible component of the special fibre is either a toric variety or a single non-toric blow up of a toric variety. Its importance will be realised in §3.8 when we compute the integral affine structure induced by the affinoid torus fibration associated to the model. The following proposition shows that we can always construct a model of this form. We give another example after the proposition to aid in understanding the construction.

Proposition 3.34. Suppose we are in Setup 3.23. Then there exists a reduced snc log model $(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})$ of $Y \setminus D$ such that each irreducible component of $\tilde{\mathcal{Y}}_k$ is either toric or a single non-toric blowup of a toric Looijenga pair (X, Δ_X) . **Can we say that it actually consists of a single toric surface and the others are all non-toric w/ charge 1. Actually, can we clean up these statements by saying we perform the birational operations until the original irreducible component has charge 0 and all the others have charge 1?**

Proof. We give an explicit construction of $\tilde{\mathcal{Y}}$ here. Write $b_i = D_i^2$ and suppose there are k_i non-toric blowups along D_i in Y .

For each divisor D_i with $k_i > 0$, we first apply Construction 3.24 *i.e.* we blow up $\mathcal{D}_i \times_R k$ in \mathcal{Y} . We then apply Construction 3.27 (i) to flop each exceptional curve whose centre is a smooth point on $\overline{\mathcal{D}}_i \times_R k$ to the newly produced irreducible component Y_1^i . Apply Construction 3.27 (ii) to all but one of the curves which are given by the strict transform of a fibre through the centre of a non-toric blowup. Denote this snc log model of $U = Y \setminus D$ by $(\mathcal{Y}', \mathcal{D}')$ where \mathcal{D}' is the strict transform of \mathcal{D} . The new irreducible component produced by applying Construction 3.24 within 3.27 (ii) is denoted Y_2^i .

Next we can run through Construction 3.27 (ii) again. Indeed, let \mathcal{D}'_i denote the strict transform of \mathcal{D}_i and apply Construction 3.24 to $\mathcal{D}'_i \times_R k$. Denote the new component produced by Y_3^i . We then flop all but one of the curves which are given by the strict transform of a fibre through the centre of a non-toric blowup. More generally, we denote the new irreducible component produced by applying Construction 3.24 to the strict transform of \mathcal{D}_i for the j^{th} time by Y_j^i . After applying Construction 3.24 for the j^{th} time to the strict transform of \mathcal{D}_i , we apply Construction 3.27 (ii)

and flop all but one of the curves which are given by the strict transform of a fibre through the centre of a non-toric blowup to the new irreducible component Y_j^i .

This ensures we leave behind one non-toric exceptional curve on Y_j^i . We can repeat this until each irreducible component of the special fibre not equal to the strict transform of Y is isomorphic to a single non-toric blowup of a toric Looijenga pair (X, Δ_X) . Recall X is a ruled toric surface as described in Proposition 3.31.

We can describe the irreducible components Y_j^i by the classification of surfaces. Indeed, we have that in the special fibre, for each i , there will be a chain of irreducible components

$$Y_1^i = \text{Bl}_{p_1} \mathbb{F}_{|b_i+k_i-1|}, Y_2^i = \text{Bl}_{p_2} \mathbb{F}_{|b_i+k_i-2|}, \dots, Y_{k_i}^i = \text{Bl}_{p_{k_i}} \mathbb{F}_{|b_i|},$$

where Y_1^i intersects Y_0 non-trivially, Y_j^i only intersects $Y_{j\pm 1}^i$ and p_i is a smooth point on the boundary of $\mathbb{F}_{|b_i+k_i-i|}$. The resulting pair is an snc log model of $U = Y \setminus D$. \square

Example 3.35. Assume we are in Setup 3.23 with (Y, D) a non-toric del Pezzo surface of degree 5 whose combinatorial type is illustrated on the left hand side of the figure below. Then the special fibre of the model arising from Proposition 3.34 is described on the right hand side of the figure below.

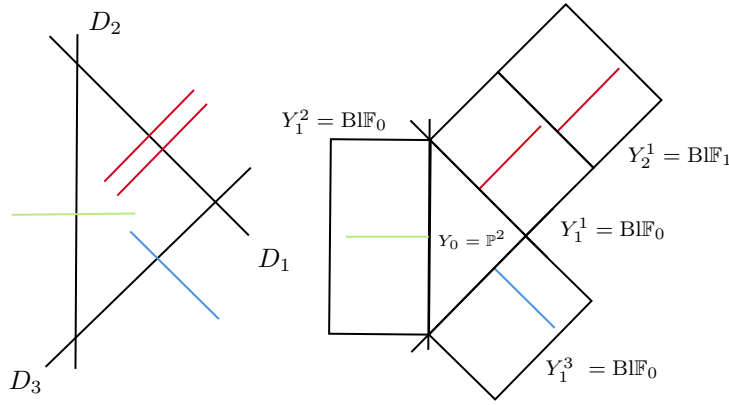


FIGURE 7. Model of non-toric dP5 using Proposition 3.34

Remark 3.36. Let $(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}})$ be the snc log model of $U = Y \setminus D$ constructed in Proposition 3.34. Then there is a 1-1 correspondence between non-toric irreducible components of $\tilde{\mathcal{Y}}_k$ and non-toric exceptional curves in (Y, D) .

Let E_j^i denote the non-toric exceptional curves with centre on \overline{D}_i for $j = 1, \dots, k_i$ and \mathcal{E}_j^i the scheme-theoretic closure of E_j^i in \mathcal{Y} . Furthermore, suppose the strict transform of \mathcal{E}_j^i , when restricted to the special fibre, cuts out the non-toric exceptional curve on Y_j^i . Then the correspondence is given by $Y_j^i \leftrightarrow E_j^i$.

3.7. Construction of the fibration. *make this higher dimensional* Let (Y, D) be a generic Looijenga pair over K satisfying Assumption 3.22 and $(\mathcal{Y}, \mathcal{D})$ a reduced snc log model of $U = Y \setminus D$ arising from the modifications considered in §3.3. By Proposition 2.19, there is a retraction map

$U^{\text{an}} \rightarrow \Sigma_{(\mathcal{Y}, \mathcal{D})}$. In this section, we prove that this map is a non-archimedean SYZ fibration (2.37) *i.e.* an affinoid torus fibration away from the 0-dimensional strata Γ of $\Sigma_{(\mathcal{Y}, \mathcal{D})}$.

Let C be a 1-dimensional stratum of \mathcal{Y}_k ; note $C \cong \mathbb{P}^1$. Let X_v denote an irreducible component of the special fibre containing C and X_w , the other prime component of $\mathcal{Y}_k + \mathcal{D}$ which contains C . Note the component X_w may be a vertical or horizontal divisor. We set

$$\begin{aligned} b_v &= \deg \mathcal{O}_C(-X_v) = -(C \cdot X_v), \\ b_w &= \deg \mathcal{O}_C(-X_w) = -(C \cdot X_w). \end{aligned}$$

Let $\mathcal{F} = (\mathcal{Y}_k \cup \mathcal{D}) \setminus (X_v \cup X_w)$ and $C^\circ = C \setminus \mathcal{F}$. The space $C \setminus C^\circ$ consists of precisely two distinct points, which we denote by c_0 and c_∞ . Let X_0 and X_∞ denote the respective prime divisors containing c_0 and c_∞ . Let N_0, N_∞, N_v, N_w denote the respective multiplicities in the special fibre \mathcal{Y}_k . By construction, $N_w, N_0, N_\infty \in \{0, 1\}$ and $N_v = 1$. Since \mathcal{Y}_k is principal, we have the following equality:

$$(2) \quad b_v N_v + b_w N_w = N_0 + N_\infty.$$

The following proposition is a special case of [NXY19, Proposition 5.4]. We remark after the proposition how the positivity assumptions can always be satisfied.

Proposition 3.37. Assume that $b_v, b_w > 0$. Then there exists a strictly semistable toric R -scheme \mathcal{X} and a stratum C' of \mathcal{X}_k such that $\widehat{\mathcal{Y}}_{/C} \cong \widehat{\mathcal{X}}_{/C'}$.

Proof. We will construct the scheme \mathcal{X} directly. Let

$$\begin{aligned} u_0 &= (1, 0, N_0) \in \mathbb{Z}^3, \\ u_w &= (0, 1, N_w) \in \mathbb{Z}^3, \\ u_v &= (0, 0, N_v) \in \mathbb{Z}^3, \\ u_\infty &= (-1, b_w, N_\infty) \in \mathbb{Z}^3. \end{aligned}$$

Let r_i be the ray spanned by u_i in $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ and consider the cones σ_0 and σ_∞ spanned by $\{r_v, r_w, r_0\}$ and $\{r_v, r_w, r_\infty\}$ respectively. Let Σ be the fan in $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ with maximal cones σ_0 and σ_∞ .

Let Σ_h denote the slice of Σ at height $h \geq 0$. By [Gub13, §7.7], the fan Σ determines a toric R -scheme \mathcal{X} whose generic fibre is isomorphic to the toric variety whose fan is Σ_0 . By [Gub13, Proposition 7.11], the special fibre \mathcal{X}_k is reduced. Moreover, since σ_0 and σ_∞ are simple cones, we have that \mathcal{X} is regular over R by [KKMSD73, §4.3]. Together, this shows that \mathcal{X} is strictly semistable.

The special fibre \mathcal{X}_k is a simple normal crossings divisor and by [Gub13, Lemma 7.10], the rays of the fan Σ correspond to the prime divisors in \mathcal{X}_k with multiplicities given by the last coordinate of the primitive generators of the rays. Thus we have the relation

$$\mathcal{X}_k = N_0 F_0 + N_\infty F_\infty + N_v F_v + N_w F_w,$$

where F_i are the irreducible components of \mathcal{X}_k with multiplicity N_i . Let $C' := F_v \cap F_w$ and let d_0, d_∞ be the respective intersections of C' with F_0 and F_∞ . By [Ful93, §5.1], we have $C' \cdot F_j = -b_j$

for $j \in \{v, w\}$. Consider the following line bundles

$$\begin{aligned}\mathcal{L}_w &= \mathcal{O}_{\mathcal{Y}}(-X_w - b_w X_\infty), \\ \mathcal{L}_v &= \mathcal{O}_{\mathcal{Y}}(-X_v - b_v X_\infty), \\ \mathcal{L}_0 &= \mathcal{O}_{\mathcal{Y}}(X_\infty - X_0), \\ \mathcal{L}_\infty &= \mathcal{L}_0^{-1}.\end{aligned}$$

By construction, the restriction of each of these line bundles to $C \cong \mathbb{P}^1$ is degree zero. Thus we can choose non-zero global sections s_w, s_v, s_0, s_∞ from the line bundles $\mathcal{L}_w|_C, \mathcal{L}_v|_C, \mathcal{L}_0|_C$ and $\mathcal{L}_\infty|_C$ respectively. By the positivity assumption on b_w and b_v , the conormal bundle $\mathcal{N}_{C/\mathcal{Y}}^\vee = \mathcal{O}_C(-X_w) \oplus \mathcal{O}_C(-X_v)$ is a direct sum of ample line bundles on C . In particular, the first cohomology group $H^1(\mathcal{N}_{C/\mathcal{Y}}^\vee)$ vanishes and we have surjective maps

$$H^0((\mathcal{Y}/C)_{n+1}, \mathcal{L}_w) \rightarrow H^0((\mathcal{Y}/C)_n, \mathcal{L}_w)$$

where $(\mathcal{Y}/C)_n$ is the n^{th} order infinitesimal thickening of C in \mathcal{Y} . The section s_w lifts to all finite order thickenings by induction and thus also lifts to a global section of $\mathcal{O}_{\widehat{\mathcal{Y}}/C}(-X_w - b_w X_\infty)$. We will continue to denote the element by s_w . Similarly, we can lift s_v, s_0 and s_∞ to the formal completion. Note $s_\infty = \frac{1}{s_0}$ and both s_0 and s_∞ are nowhere vanishing global sections of \mathcal{L}_0 and \mathcal{L}_∞ on $\widehat{\mathcal{Y}}/C$ respectively.

We now construct the isomorphism $f : \widehat{\mathcal{Y}}/C \rightarrow \widehat{\mathcal{X}}/C'$ by describing it on local toric charts and showing they agree on the intersections. To do this, consider the following open formal subschemes:

$$\mathcal{Y}_0 = \widehat{\mathcal{Y}}/C \setminus \{c_\infty\}, \quad \mathcal{Y}_\infty = \widehat{\mathcal{Y}}/C \setminus \{c_0\}, \quad \mathcal{X}_0 = \widehat{\mathcal{X}}/C' \setminus \{d_\infty\}, \quad \mathcal{X}_\infty = \widehat{\mathcal{X}}/C' \setminus \{d_0\}.$$

By construction, since s_w is a global section of $\mathcal{O}_{\widehat{\mathcal{Y}}/C}(-X_w - b_w X_\infty)$, we have that s_w is a global equation for X_w in \mathcal{Y}_0 . Similarly, s_v is a global equation for X_v in \mathcal{Y}_0 and s_0 is a global equation for X_0 in \mathcal{Y}_0 . On \mathcal{Y}_∞ , we have that $s_w s_\infty^{-b_w}$ and $s_v s_\infty^{-b_v}$ define X_w and X_v respectively whilst s_∞ defines X_∞ . The element $\omega = t s_0^{-N_0} s_w^{-N_w} s_v^{-N_v} \in \mathcal{O}_{\widehat{\mathcal{Y}}/C}$ defines an invertible regular function on $\widehat{\mathcal{Y}}/C$ by (2).

Let $\{u_0^\vee, u_v^\vee, u_w^\vee\}$ be the dual basis to $\{u_0, u_v, u_w\}$. Then we have

$$(3) \quad \mathcal{X}_0 = \text{Spf } R\{\chi^{u_0^\vee}\}[[\chi^{u_w^\vee}, \chi^{u_v^\vee}]] / \left(t - \chi^{N_0 u_0^\vee + N_w u_w^\vee + N_v u_v^\vee}\right).$$

Choose integers $\alpha_0, \alpha_w, \alpha_v$ such that $\alpha N_0 + \alpha_w N_w + \alpha_v N_v = 1$ and define the morphism $f_0 : \mathcal{Y}_0 \rightarrow \mathcal{X}_0$ of formal R -schemes by the following morphism of topological R -algebras:

$$\begin{aligned}\mathcal{O}(\mathcal{X}_0) &\longrightarrow \mathcal{O}(\mathcal{Y}_0) \\ \chi^{u_i^\vee} &\longrightarrow \omega^{\alpha_i} s_i.\end{aligned}$$

By employing the same argument as in [NXY19, Proposition 5.4], f_0 is an isomorphism.

We next consider the pair of formal toric charts \mathcal{Y}_∞ and \mathcal{X}_∞ . The lattice vectors $\{-u_0^\vee, u_w^\vee + b_w u_0^\vee, u_v^\vee + b_v u_0^\vee\}$ are a dual basis to $\{u_\infty, u_w, u_v\}$ and we have

$$(4) \quad \mathcal{X}_\infty = \text{Spf } R\{\chi^{-u_0^\vee}\}[[\chi^{u_w^\vee + b_w u_0^\vee}, \chi^{u_v^\vee + b_v u_0^\vee}]] / \left(t - \chi^{N_0 u_0^\vee + N_w u_w^\vee + N_v u_v^\vee}\right).$$

Define the morphism $f_\infty : \mathcal{O}(\mathcal{X}_\infty) \rightarrow \mathcal{O}(\mathcal{Y}_\infty)$ as

$$\begin{aligned} \chi^{-u_0^\vee} &\mapsto \omega^{-\alpha_0} s_\infty \\ \chi^{u_w^\vee + b_w u_0^\vee} &\mapsto \omega^{\alpha_w + b_w \alpha_0} s_w s_\infty^{-b_w} \\ \chi^{u_v^\vee + b_v u_0^\vee} &\mapsto \omega^{\alpha_v + b_v \alpha_0} s_v s_\infty^{-b_v}. \end{aligned}$$

By the same reasoning, f_∞ is an isomorphism. On the intersections $\mathcal{Y}_{0,\infty} := \mathcal{Y}_0 \cap \mathcal{Y}_\infty$ and $\mathcal{X}_{0,\infty} := \mathcal{X}_0 \cap \mathcal{X}_\infty$, the maps f_0 and f_∞ agree and thus they both glue to give an isomorphism of formal schemes

$$f : \widehat{\mathcal{Y}}/C \rightarrow \widehat{\mathcal{X}}/C'.$$

□

Remark 3.38. In the statement of Proposition 3.37, we made the assumption that $b_v, b_w > 0$ to ensure that the algebraic sections s_j lifted to formal sections. By performing toric blowups at 0-dimensional strata of \mathcal{Y} , we can always satisfy this positivity assumption. This may destroy the property that \mathcal{Y}_k is reduced but the special fibre of the model will still remain snc in the total space.

Remark 3.39. Let X be an irreducible component of \mathcal{Y}_k with boundary $\Delta_X = \sum_{i=1}^k \Delta_{X,i}$ and v_X the corresponding 0-dimensional stratum of $\Sigma_{(\mathcal{Y}, \mathcal{D})}$. By adjunction, the pair (X, Δ_X) is a log Calabi-Yau surface over k . In [EM21, Theorem 6.14], the authors construct a Lagrangian torus fibration

$$\phi : \mathcal{U} \rightarrow B \subseteq \text{Star}(v_X) \setminus V,$$

where \mathcal{U} is a symplectic tubular neighbourhood of Δ_X , B a retract of $\text{Star}(v_X) \setminus V$ and V the union of strata of codimension greater than or equal to 2 in $\Sigma_{(\mathcal{Y}, \mathcal{D})}$.

The fibration is constructed by gluing toric moment maps which are defined in a neighbourhood of each component of Δ_X . The integral affine structure induced by ϕ on B is dual to the integral affine structure induced by the non-archimedean SYZ fibration $\rho_{(\mathcal{Y}, \mathcal{D})}$. In particular, the integral affine structure arising from non-archimedean geometry has a symplectic topological analogue.

Remark 3.40. In the proof of Proposition 3.37, we showed that the elements s_0, s_w, s_v and s_∞ are coordinates, satisfying some relations, on the preimage of $\text{Star}(\tau_C)$ under the retraction map $\rho_{(\mathcal{Y}, \mathcal{D})}$. Indeed, by Example 2.20, by taking analytic generic fibres of the open formal schemes (3) and (4), we see that the pair (s_0, s_w) form a pair of affinoid coordinates on $\rho^{-1}(\text{Star}(\tau_C)) =]C[\subseteq U^{\text{an}}$.

We can now prove that the associated retraction map is a non-archimedean SYZ fibration.

Theorem 3.41. Let (Y, D) be a generic Looijenga pair satisfying Assumption 3.22 and $(\mathcal{Y}, \mathcal{D})$ a reduced snc log model of $U = Y \setminus D$ arising from the modifications considered in §3.3. Let Γ be the set of vertices of $\Sigma_{(\mathcal{Y}, \mathcal{D})}$. Then the retraction map $\rho_{(\mathcal{Y}, \mathcal{D})} : U^{\text{an}} \rightarrow \Sigma_{(\mathcal{Y}, \mathcal{D})}$ is an affinoid torus fibration over $\Sigma_{(\mathcal{Y}, \mathcal{D})} \setminus \Gamma$. In particular, $\rho_{(\mathcal{Y}, \mathcal{D})}$ is a non-archimedean SYZ fibration.

Proof. We first show that $\rho_{(\mathcal{Y}, \mathcal{D})}$ is an affinoid torus fibration over the top dimensional open faces of $\Sigma_{(\mathcal{Y}, \mathcal{D})}$. Let $\mathring{\sigma}$ be such an open face - this corresponds to a 0-dimensional stratum y of the special fibre \mathcal{Y}_k i.e. $\mathring{\sigma} = \text{Star}(\tau_y)$. By Example 2.20, we know $\rho_{(\mathcal{Y}, \mathcal{D})}^{-1}(\text{Star}(\tau_y))$ is isomorphic to the analytic generic fibre of the formal scheme $\text{Spf } \widehat{\mathcal{O}}_{\mathcal{Y}, y}$. Since \mathcal{Y}_k is reduced and $(\mathcal{Y}, \mathcal{D})$ an snc pair, we have

$$\widehat{\mathcal{O}}_{\mathcal{Y}, y} \cong R[[y_0, y_1, y_2]] / (y_0 y_1 y_2 - t).$$

By Example 2.34, we can identify the restriction of $\rho_{(\mathcal{Y}, \mathcal{D})}$ over $\text{Star}(\tau_y)$ with the restriction of the tropicalisation map

$$\rho_{\mathbb{G}_m^{2, \text{an}}} : \mathbb{G}_m^{2, \text{an}} \longrightarrow \mathbb{R}^2$$

over the 2-dimensional open simplex

$$\{(v_0, v_1, v_2) \in \mathbb{R}_{>0}^2 \mid v_0 + v_1 + v_2 = 1\}.$$

It follows that over $\text{Star}(\tau_y)$, $\rho_{(\mathcal{Y}, \mathcal{D})}$ is a 2-dimensional affinoid torus fibration.

It remains to extend the fibration over the 1-dimensional strata in $\Sigma_{(\mathcal{Y}, \mathcal{D})}$. These each correspond to a 1-dimensional stratum C of the special fibre \mathcal{Y}_k . Let $\hat{\tau}_C$ denote an open 1-dimensional stratum in $\Sigma_{(\mathcal{Y}, \mathcal{D})}$. Using the notation of Proposition 3.37, we can perform toric blowups at 0-dimensional strata to ensure that $b_w, b_v > 0$ and we can further assume $N_0, N_\infty > 0$. By [MN15, Proposition 3.1.7], this does not affect $\rho_{(\mathcal{Y}, \mathcal{D})}$. The effect on the skeleton $\Sigma_{(\mathcal{Y}, \mathcal{D})}$ is a sequence of star subdivisions of the top dimensional faces corresponding to the zero-dimensional strata that are blown up [MN15, 3.1.9]. This does not affect $\hat{\tau}_C$ and, since $\rho_{(\mathcal{Y}, \mathcal{D})}$ is a 2-dimensional affinoid torus fibration over top dimensional faces, we can assume the positivity assumption is satisfied. Similarly, by a toric blowup at C in \mathcal{Y} , we can also assume $N_w > 0$.

Consider the subset $\text{Star}(\hat{\tau}_C)$ - this is equal to the union of the open faces corresponding to the strata c_0, c_∞ and C in \mathcal{Y}_k . Let \mathcal{X} be the toric R -scheme associated to the fan Σ constructed in the proof of Proposition 3.37 with the closure of a 1-dimensional torus orbit C' corresponding to $\sigma_0 \cap \sigma_\infty$ and $\hat{\mathcal{Y}}_{/C} \cong \hat{\mathcal{X}}_{/C'}$.

Now let $V = \Sigma_1$ be the height 1 slice of Σ . Then the restriction of $\rho_{(\mathcal{Y}, \mathcal{D})}$ over $\text{Star}(\hat{\tau}_C)$ can be identified with the restriction of the 2-dimensional affinoid torus fibration $\rho_{\mathbb{G}_m^{2, \text{an}}}$ over V . Thus the fibration extends to an affinoid torus fibration over all codimension 1 strata of $\Sigma_{(\mathcal{Y}, \mathcal{D})}$. \square

Definition 3.42. Let (Y, D) be a generic Looijenga pair satisfying Assumption 3.22. When $(\mathcal{Y}, \mathcal{D})$ is the snc log model of U constructed in Proposition 3.34, we denote the skeleton $\Sigma_{(\mathcal{Y}, \mathcal{D})}$ by $\text{Sk}(U)$.

3.8. Integral affine structure. In the previous section, we showed that $\rho_{(\mathcal{Y}, \mathcal{D})}$ is an affinoid torus fibration away from the set Γ of codimension 2 strata. In this section, we describe the induced integral affine structure on $\text{Sk}(U) \setminus \Gamma$. We will denote the snc log model of $U = Y \setminus D$ from Proposition 3.34 by $(\mathcal{Y}, \mathcal{D})$. In particular, we compute the monodromy of the integral affine structure around each of the vertices in $\text{Sk}(U)$. We will use Proposition 3.37 which gave us an explicit description of the local toric charts. We first show that the integral affine structure is determined by the intersection numbers of the 1-dimensional strata in the special fibre of \mathcal{Y} . Whilst we will only carry out this computation for the model constructed in Proposition 3.34, the computation is similar for the general models considered in §3.3.

We start by recalling how the integral affine structure arises from the affinoid torus fibration. Following the notation of §3.7, let $C = X_w \cap X_v$ be a 1-dimensional stratum of \mathcal{Y}_k and denote by τ_C the corresponding 1-dimensional stratum in $\text{Sk}(U)$. The open star of τ_C consists of two maximal faces corresponding to the 0-dimensional stratum $c_0 = C \cap X_0$ and $c_\infty = C \cap X_\infty$ joined along τ_C . Let \mathcal{X} denote the local toric model over R associated to the fan Σ with C' the corresponding 1-dimensional stratum constructed in the proof of Proposition 3.37 such that $\hat{\mathcal{Y}}_{/C} \cong \hat{\mathcal{X}}_{/C'}$. Then by Example 2.34 there is a commutative diagram

$$\begin{array}{ccc}]C[& \xrightarrow{\sim} &]C'[\\ \downarrow \rho_{(\mathcal{Y}, \mathcal{D})} & & \downarrow \text{trop} \\ \text{Star}(\tau_C) & \xrightarrow[\phi]{\sim} & \Sigma_1 \end{array}$$

where the upper arrow is an isomorphism of K -analytic spaces and the lower one a homeomorphism. The K -analytic spaces $]C[$ and $]C'[$ are the analytic generic fibres of $\widehat{\mathcal{Y}}/C$ and $\widehat{\mathcal{X}}/C'$ respectively. By definition, the integral affine structure on $\text{Star}(\tau_C)$ is the pullback of the integral affine structure on Σ_1 .

Let v_X be a vertex of $\Sigma_{(\mathcal{Y}, \mathcal{D})}$ corresponding to the irreducible component X in the special fibre \mathcal{Y}_k with boundary $\Delta_X = \sum_{i=1}^k \Delta_{X,i}$. Let $(\mathcal{X}, \Delta_{\mathcal{X}}) = (X, \Delta_X) \times_k R$ denote the constant family with fibre (X, Δ_X) . By the proof of Proposition 3.41, the retraction map $(X \setminus \Delta_X)^{\text{an}} \rightarrow \Sigma_{(\mathcal{X}, \Delta_{\mathcal{X}})}$ is also an affinoid torus fibration away from the vertex $O \in \Sigma_{\mathcal{X}, \Delta_{\mathcal{X}}}$. Thus the skeleton $\Sigma_{(\mathcal{X}, \Delta_{\mathcal{X}})}$ is an integral affine manifold away from O .

Lemma 3.43. There is a canonical identification

$$\varphi_v : \text{Star}(v_X) \xrightarrow{\sim} \Sigma_{(\mathcal{X}, \Delta_{\mathcal{X}})}$$

of singular integral affine manifolds.

Proof. By construction, the polyhedral decompositions of $\text{Star}(v)$ and $\Sigma_{(\mathcal{X}, \Delta_{\mathcal{X}})}$ are the same. By the proof of Proposition 3.37 and the discussion above, the integral affine structures around v and O are both given by the intersection numbers of each of the 1-dimensional strata C in the boundary of X with the prime divisor not containing C - by construction, these numbers match up and are given by the self intersection numbers in X . Thus $\text{Star}(v_X)$ and $\Sigma_{(\mathcal{X}, \Delta_{\mathcal{X}})}$ are canonically identified. \square

By Lemma 3.43, we have that the induced singular integral affine structure around v_X induced by the non-archimedean SYZ fibration coincides with the singular integral affine structure on $\Sigma_{(\mathcal{X}, \Delta_{\mathcal{X}})}$. By [GHK15a, Lemma 1.3], the integral affine structure on $\Sigma_{(\mathcal{X}, \Delta_{\mathcal{X}})} \setminus \{O\}$ extends across the origin if and only if X is toric and Δ_X is the toric boundary. Thus we have the following corollary:

Corollary 3.44. Let Y_0 denote the toric irreducible component of \mathcal{Y}_k . Then the integral affine structure extends across v_{Y_0} . We will denote the vertex v_{Y_0} by O .

Let X denote an irreducible component of the special fibre $\mathcal{Y}_k = X + \sum_{i=1}^r X_i$ and v_X the corresponding vertex in $\text{Sk}(U)$. Let $\Delta_X = \sum_{i=1}^r X_i \cap X + \mathcal{D} \cap X$. By the description of $(\mathcal{Y}, \mathcal{D})$ in Proposition 3.34, we know X is either toric or isomorphic to a non-toric blowup of \mathbb{F}_n . In the toric case, the integral structure will extend across the vertex by Corollary 3.44. In the latter case, we know $\Delta_X = \sum_{j=1}^4 C_j$. Let τ_{C_i} denote the corresponding 1-dimensional stratum in $\text{Sk}(U)$. Since $\text{Sk}(U) \subset \mathbb{R}^{I^v \cup I^h}$, let $v_{C_i} \in \mathbb{Z}^{I^v \cup I^h}$ denote the primitive generator of the ray defining τ_{C_i} .

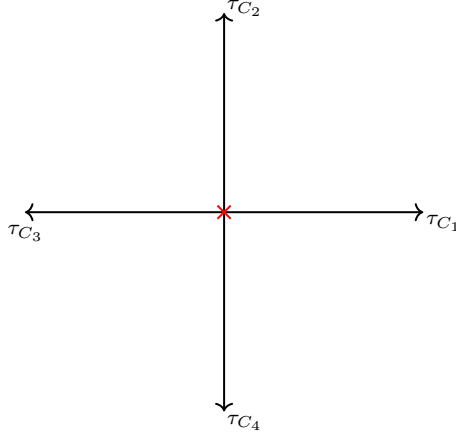


FIGURE 8. Local singularity

Proposition 3.45. The monodromy $T_{\rho(\mathcal{Y}, \mathcal{D})}^{\text{int}}$ of the integral affine structure induced by $\rho(\mathcal{Y}, \mathcal{D})$ around v_X is

$$T_{\rho(\mathcal{Y}, \mathcal{D})}^{\text{int}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with respect to the basis $(\chi^{v_{C_1}^\vee}, \chi^{v_{C_2}^\vee})$.

Proof. Let $b_i = -(C_i \cdot H) = -(C_i^2)_X$ where $H \neq X$ is the prime component of \mathcal{Y} containing C_i . Further assume the centre of the non-toric blowup is on C_1 . The integral affine structure on $\text{Star}(\tau_{C_i})$ identifies the set $(v_{C_{i-1}}, v_{C_i}, v_X, v_{C_{i+1}})$ with

$$(v_0 = (1, 0), v_1 = (0, 1), v_2 = (0, 0), v_\infty = (-1, b_i)),$$

while the integral affine structure on $\text{Star}(\tau_{C_{i+1}})$ identifies the set $(v_{C_i}, v_{C_{i+1}}, v_X, v_{C_{i+2}})$ with

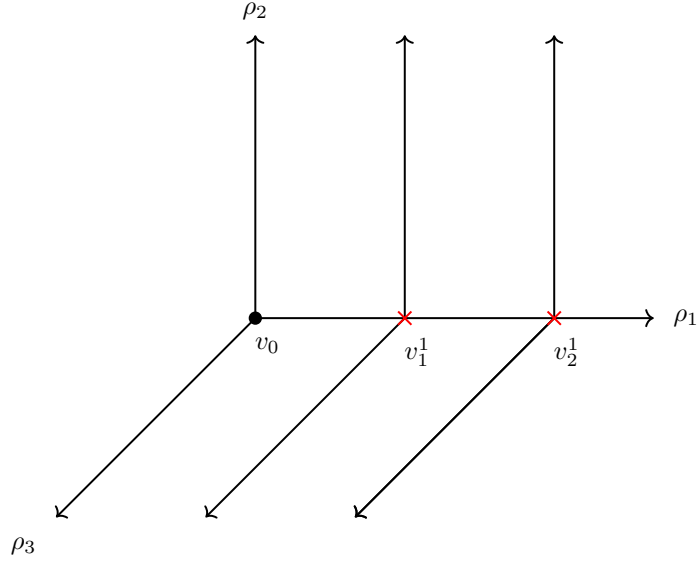
$$(v'_0 = (1, 0), v'_1 = (0, 1), v'_2 = (0, 0), v'_\infty = (-1, b_{i+1})).$$

Thus the transition map for the tangent bundle Λ from the chart $\text{Star}(\tau_{C_i})$ to the chart $\text{Star}(\tau_{C_{i+1}})$ on the intersection $\text{Star}(\tau_{C_i}) \cap \text{Star}(\tau_{C_{i+1}})$ is given by the matrix

$$\begin{pmatrix} b_i & 1 \\ -1 & 0 \end{pmatrix}$$

with respect to the basis $(v_{C_{i-1}}, v_{C_i})$. The transition function for the cotangent bundle $\check{\Lambda}$ is given by the same matrix with respect to the basis $(\chi^{v_{C_{i-1}}^\vee}, \chi^{v_{C_i}^\vee})$. The composition of these matrices then gives the monodromy. Indeed, take a loop around v_X connecting $v_{C_1}, v_{C_2}, v_{C_3}, v_{C_4}$ and v_{C_1} in that order. Then the monodromy around this loop with respect to the basis $(\chi^{v_{C_1}^\vee}, \chi^{v_{C_2}^\vee})$ is

$$\begin{pmatrix} b_1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_4 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b_2 & 1 \\ -1 & 0 \end{pmatrix}.$$



By our assumption on C_1 and the geometry of the model $(\mathcal{Y}, \mathcal{D})$, we have $b_2 = b_4 = 0$ and $b_1 + b_3 = -1$. Thus the composition of matrices above is equal to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This completes the proof. \square

Notation 3.46. Given any simply connected subset $\tau \subset \text{Sk}(U)^{\text{sm}}$, we write $\check{\Lambda}_\tau$ for the stalk of $\check{\Lambda}$ on $\text{Sk}(U)^{\text{sm}}$ at any point of τ ; any two such stalks are canonically identified by parallel transport. Given an edge/ray of ρ of $\text{Sk}(U)$, we have $\check{\Lambda}_\rho \cong \mathbb{Z}$.

Corollary 3.47. The invariant cotangent direction is parallel to $\check{\Lambda}_{\tau_{C_1}}$.

In Lemma 3.32, we described the simplicial structure of $\text{Sk}(U)$. In particular, the singular vertices of $\text{Sk}(U)$ which correspond to the non-toric exceptional curves which have a centre on \overline{D}_i can be viewed as lying on a ray ρ_i supporting the origin v_0 (corresponding to the toric irreducible component) and the vertices $v_1^i, \dots, v_{k_i}^i$. We depict this below for Example 3.33.

The same computation as in Proposition 3.45 for the monodromy of the integral affine structure around a loop containing multiple singular vertices follows readily. Indeed, suppose $v_X = v_j^i$ and consider a loop around $v_X = v_j^i, v_{j+1}^i, \dots, v_k^i$. Then the monodromy is

$$\begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$$

with respect to the coordinates $(\chi^{v_{C_1}^\vee}, \chi^{v_{C_2}^\vee})$ described in Proposition 3.45. Thus the invariant cotangent directions for the monodromy around v_j^i and v_{j+1}^i agree on $\text{Star}(v_j^i)$ and $\text{Star}(v_{j+1}^i)$. Let $S = \cup_{j=1}^{k_i} \text{Star}(v_j^i)$. Then with Proposition 3.45, we have that $\check{\Lambda}_{\tau_{C_1}} \cong \mathbb{Z}\langle \chi^{v_{C_1}^\vee} \rangle$ and $\check{\Lambda}|_S \cong \mathbb{Z}$. By parallel transport, $\chi^{v_{C_1}^\vee}$ can be extended trivially to all of S . We can then define $\check{\Lambda}_{\rho_i} = \check{\Lambda}_{\tau_{C_1}}$ for any singular vertex on ρ_i .

Notation 3.48. Let $(\mathcal{Y}, \mathcal{D})$ denote the snc log model of $U = Y \setminus D$ constructed in Proposition 3.34. From now on, let Γ be the set of singular vertices corresponding to non-toric irreducible components of \mathcal{Y}_k and let $\mathrm{Sk}(U)^{\mathrm{sm}} = \mathrm{Sk}(U) \setminus \Gamma$. In particular, $\mathrm{Sk}(U)^{\mathrm{sm}}$ now contains v_0 since the integral affine structure arising from $\rho_{(\mathcal{Y}, \mathcal{D})}$ extends across the toric vertex v_0 .

4. TROPICAL GEOMETRY

Let (Y, D) be a Looijenga pair. Recall the following definition from §3.2:

$$D^\perp := \{\alpha \in \mathrm{Pic}(Y) \mid \alpha \cdot [D_i] = 0 \text{ for all } i\} \subseteq \mathrm{Pic}(Y).$$

Let $\iota : \mathrm{Sk}(U)^{\mathrm{sm}} \rightarrow \mathrm{Sk}(U)$ denote the embedding of the smooth locus $\mathrm{Sk}(U)^{\mathrm{sm}}$ of the integral affine structure into $\mathrm{Sk}(U)$. In this chapter, we will prove a tropical correspondence result (4.35) between D^\perp and $H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})_{\mathrm{tf}}$ **what is tf?**. The construction of the model $(\mathcal{Y}, \mathcal{D})$ from Proposition 3.34 and its associated skeleton is essential for this correspondence. To do this, we will introduce two objects, called tropical spokes and tropical wings. Their definition is purely combinatorial and reflects the notion of Symington's relative cycles in [Sym] *i.e.* elements of the relative homology group $H_1(\mathrm{Sk}(U), \Gamma : \iota_* \check{\Lambda})$. Moreover, they closely resemble the tropicalisations of algebraic cycles in D^\perp (4.24) arising in toric and tropical geometry. The proof of the correspondence passes through the space of these objects - the calculus of tropical spokes and wings affords us a very simple description of a basis of $H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})_{\mathrm{tf}}$.

We give an overview of the contents of this chapter. In §4.1, we introduce the notion of tropical spokes and wings and their topological realisations and we prove several results regarding their group structure. In §4.2 we describe the process of constructing tropical spokes and wings from elements of D^\perp which we will refer to as *tropicalisation* and prove an equivalence (4.27) between D^\perp and the group of tropical spokes and wings. We discuss in Remark 4.24 how to compute the tropicalisation of an element in D^\perp arising from an irreducible curve in Y using the non-archimedean SYZ fibration. In §4.3, we introduce tropical cycles and give a local description around a singular vertex of $\mathrm{Sk}(U)$ (4.29). We then complete the proof of the tropical correspondence (4.35) between D^\perp and $H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})_{\mathrm{tf}}$ and describe an explicit basis of tropical cycles on $\mathrm{Sk}(U)$.

We start by giving some preliminary definitions and results concerning D^\perp .

Definition 4.1. The charge $Q(Y, D)$ of a Looijenga pair (Y, D) is

$$Q(Y, D) = 12 - D^2 - r(D)$$

where $r(D)$ is the number of irreducible components of D .

Remark 4.2. The charge is an invariant intrinsic to the log Calabi-Yau surface. When (Y, D) is a Looijenga pair over \mathbb{C} , we have the following equalities [Fri16]:

$$Q(Y, D) = 2 + b_2(Y) - r(D) = \chi_{\mathrm{top}}(U),$$

where χ_{top} denotes the topological Euler characteristic.

Lemma 4.3. [Fri16, Lemma 2.7] The charge satisfies $Q(Y, D) \geq 0$ with equality if and only if (Y, D) is toric.

Lemma 4.4. The subspace D^\perp is free of rank $Q(Y, D) - 2 + s$ where s is the rank of the kernel of $\bigoplus_i \mathbb{Z}[D_i] \rightarrow \mathrm{Pic}(Y)$.

Proof. Since Y is a smooth rational surface, $\text{Pic}(Y)$ is a lattice and thus the subspace $D^\perp \subseteq \text{Pic}(Y)$ is free. The rank computation follows from the exact sequence

$$0 \rightarrow \mathbb{Q}^s \rightarrow \bigoplus \mathbb{Q}[D_i] \rightarrow H^2(Y, \mathbb{Q}) \rightarrow D^\perp \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow 0$$

□

In particular, D^\perp precisely encodes the non-toric nature of the geometry.

From now on, (Y, D) will be a generic Looijenga pair satisfying Assumption 3.22 and $\text{Sk}(U) = \Sigma_{(\mathcal{Y}, \mathcal{D})}$ where $(\mathcal{Y}, \mathcal{D})$ is the model constructed in Proposition 3.34 and $\overline{\text{Sk}}(U) = \overline{\Sigma}_{(\mathcal{Y}, \mathcal{D})}$. We fix some notation for the rest of this chapter.

Notation 4.5. Recall the skeleton comes equipped with a polyhedral decomposition and denote the toric vertex by v_0 of $\text{Sk}(U)$ as O . Let ρ_i denote the ray in $\text{Sk}(U)$ emanating from O which supports the singular vertices v_j^i of $\text{Sk}(U)$ corresponding to non-toric blowups with centre on \overline{D}_i as described in the previous section. We will use the convention that $1 \in \mathbb{Z} \cong \check{\Lambda}_{\rho_i}$ corresponds to the integral cotangent vector which pairs positively with the tangent vector in the direction of ρ_i i.e. pointing away from O .

Along ρ_i in $\text{Sk}(U)$, there is a natural ordering of the vertices v_j^i given by the distance from O . We will write $v_j^i < v_k^i$ to denote that the distance of v_k^i from O is greater than the distance of v_j^i from O . Note that $v_j^i < v_k^i$ if and only if $j < k$. Denote the (collection of) edge(s) connecting O and v_j^i in ρ_i by τ_j^i and let $\tau_{j,j+1}^i$ be the edge connecting v_j^i and v_{j+1}^i in ρ_i .

Let $V(\overline{\text{Sk}}(U))$ be the set of vertices of $\overline{\text{Sk}}(U)$ and $E(\text{Sk}(U))$ the set of finite edges in $\text{Sk}(U)$. Given a vertex $v \in V(\overline{\text{Sk}}(U))$, let $[O, v]$ denote the edge/ray connecting O and v and let $\check{e}_v \in \check{\Lambda}_{[O, v]}$ denote the primitive cotangent vector corresponding to 1 under the isomorphism above.

4.1. Tropical spokes and wings.

Definition 4.6. A *tropical spoke* is a function

$$f : V(\overline{\text{Sk}}(U)) \setminus \{O\} \rightarrow \mathbb{Z}$$

which satisfies the balancing condition $\sum_{v \in V(\overline{\text{Sk}}(U)) \setminus O} f(v) \check{e}_v = 0$. We say that f is an *infinite* tropical spoke if $f(v) = 0$ for all vertices in $\text{Sk}(U) \setminus \{O\}$ and *finite* if $f(v) = 0$ for all vertices that lie at infinity in $\overline{\text{Sk}}(U)$. Let ∞_i denote the vertex in $\overline{\text{Sk}}(U) \setminus \text{Sk}(U)$ given by the closure of ρ_i . We say that a tropical spoke is *realisable* if for each ray ρ_i , there is at most one vertex $v \in \{v_1^i, \dots, v_{k_i}^i, \infty_i\}$ such that $f(v) \neq 0$ and $f(v) = 0$ if v does not lie in the closure of ρ_i in $\overline{\text{Sk}}(U)$ for some i .

Definition 4.7. The *topological realisation* τ_f of a realisable tropical spoke f is the subset of $\overline{\text{Sk}}(U)$ given by

$$f(v) \neq 0 \iff [O, v] \subset \tau_f$$

with the edge/ray $[O, v]$ equipped with the weight vector $f(v) \check{e}_v \in \check{\Lambda}_{[O, v]}$.

Definition 4.8. We say that a finite tropical spoke f is *primitive* if

- (i) it is realisable with $f(v_{j_i}^i) \neq 0$;
- (ii) the tuple of weights $(f(v_{j_i}^i))_i$ is coprime;
- (iii) no strict subset of $\{v_{j_i}^i : i\}$ satisfies the balancing condition.

We will refer to a primitive tropical spoke as *simple* if for each i with $v_{j_i}^i$ such that $f(v_{j_i}^i) \neq 0$, we have $v_{j_i}^i = v_1^i$.

Denote the set of finite tropical spokes as $\underline{\text{TropSpoke}}$. The set $\underline{\text{TropSpoke}}$ can be endowed with a natural abelian group structure given by addition.

Example 4.9. Let Y be \mathbb{P}^2 blown up non-torically once along each boundary divisor and $D = D_1 + D_2 + D_3$ be the strict transform of the toric boundary divisor. Let L denote the pullback of the line class on \mathbb{P}^2 and E_i denote the exceptional curve from blowing up $p_i \in D_i^\circ$. Then $[L] - [E_1] - [E_2] - [E_3] \in D^\perp$ and it is represented by an irreducible (-2) -curve C in $U = Y \setminus D$. There is also exactly one primitive tropical spoke (up to sign); its topological realisation is the finite graph supported on the 1-strata of the fan of \mathbb{P}^2 depicted below.

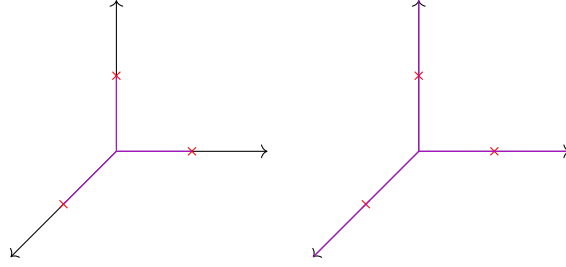


FIGURE 9. Finite and infinite tropical spoke for non-toric dP6. The left hand spoke is primitive.

Definition 4.10. A *tropical wing* is a function

$$f : E(\text{Sk}(U)) \setminus E(\text{Sk}(U))_O \rightarrow \mathbb{Z}$$

where $E(\text{Sk}(U))_O$ are the finite edges of $\text{Sk}(U)$ with O as a vertex. A tropical wing f is *realisable* if for each ray ρ_i , there exists at most one edge $\tau_{j,j+1}^i \subset \rho_i$ such that $f(\tau_{j,j+1}^i) \neq 0$ and for all other edges e supported on ρ_i , we have $f(e) = 0$.

Definition 4.11. The *topological realisation* τ_f of a realisable tropical wing f is the subset of $\text{Sk}(U)$ such that

$$f(\tau_{j,j+1}^i) \neq 0 \iff \tau_{j,j+1}^i \subset \tau_f$$

equipped with the weight covector $f(\tau_{j,j+1}^i) \in \mathbb{Z} \cong \check{\Lambda}_{\rho_i}$.

Definition 4.12. A tropical wing is *simple* if there exist unique i and j such that $f(\tau_{j,j+1}^i) = \pm 1$ and $f(\tau_{j',j'+1}^i) = 0$ otherwise. We say a primitive tropical wing is *positively oriented* if $f(\tau_{j,j+1}^i) = 1$ and *negatively oriented* if $f(\tau_{j,j+1}^i) = -1$.

Example 4.13. Consider \mathbb{P}^2 blown up at 4 points in the configuration in Figure 11. Then D^\perp is a rank 2 lattice generated by $\alpha = [L] - [E_1] - [E_2] - [E_3]$ and $\beta = [E_1] - [E_4]$ where E_1 and E_4 are the orange and blue exceptional curves respectively. The tropicalisation of α is a tropical spoke (in yellow) and β is a tropical wing (in purple) as drawn on the right hand side.

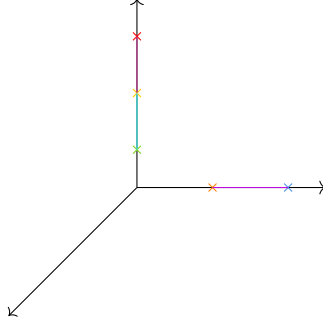


FIGURE 10. Examples of realisable tropical wings

Denote the set of tropical wings as $\underline{\text{TropWing}}$. There is a natural abelian group structure on $\underline{\text{TropWing}}$ given by addition. We then have the following easy lemma:

Lemma 4.14. There is a group isomorphism

$$\underline{\text{TropWing}} \longrightarrow \prod_{i:k_i \neq 0} \mathbb{Z}^{k_i-1}.$$

Proof. This is clear from the the definition of tropical wings. \square

Lemma 4.15. Suppose there are $k \geq 3$ singularities in $\text{Sk}(U)$. Then there is a group isomorphism

$$\underline{\text{TropSpoke}} \longrightarrow \mathbb{Z}^{k-2}.$$

Proof. Note that for $k \geq 3$, the balancing condition of tropical spokes defines a codimension 2 sublattice of \mathbb{Z}^k and thus we are done. \square

Observe that the primitive tropical spokes are the generators of $\underline{\text{TropSpoke}}$.

Corollary 4.16. Suppose that there are $k \geq 3$ singularities in $\text{Sk}(U)$ and exactly q rays have singularities. Then there are exactly $q - 2$ simple tropical spokes.

Proof. We have $\sum_i k_i = k$ and $q + \sum_{i:k_i \neq 0} (k_i - 1) = k$. The condition of being a simple tropical spoke imposes $\sum_{i:k_i \neq 0} (k_i - 1)$ independent relations and thus defines a codimension $\sum_{i:k_i \neq 0} (k_i - 1)$ lattice of \mathbb{Z}^{k-2} . This defines a sublattice of \mathbb{Z}^{k-2} of rank $k - 2 - \sum_{i:k_i \neq 0} (k_i - 1) = q - 2$ whose generators are precisely the simple tropical spokes. \square

We will often use the notion of tropical spokes/wings interchangeably with their topological realisations; we will refer to such tropical spokes/wings as *realised*.

4.2. Tropicalisation. We now construct a homomorphism

$$\text{Trop} : D^\perp \longrightarrow \underline{\text{TropSpoke}} \oplus \underline{\text{TropWing}}$$

which is an analogue of the tropicalisation map in tropical and toric geometry. Note we can decompose $\text{Pic}(Y)$ as follows:

$$\text{Pic}(Y) = \text{Pic}(\overline{Y}) \oplus \bigoplus_{i,j} \mathbb{Z}[E_j^i]$$

where E_j^i is an exceptional divisor with centre on D_i and $\text{Pic}(\overline{Y})$ is the pullback of the Picard group of the toric model \overline{Y} . This is called a *marking* of the Picard group of Y . Furthermore, recall that E_j^i corresponds to the irreducible component Y_j^i . Since $\text{Pic}(\overline{Y})$ is generated by the \overline{D}_i for $i = 1, \dots, \ell(D)$, we can write every $\alpha \in D^\perp$ as

$$\alpha = \alpha_{\mathbf{t}} + \alpha_{\mathbf{nt}}$$

where $\alpha_{\mathbf{t}} \in \bigoplus_i \mathbb{Z}[D_i]$ and $\alpha_{\mathbf{nt}} \in \bigoplus_{i,j} \mathbb{Z}[E_j^i]$.

Remark 4.17. Note that $\alpha_{\mathbf{t}} \cdot \alpha_{\mathbf{nt}}$ is not necessarily 0 though there exists an orthogonal decomposition of $\text{Pic}(Y) = \text{Pic}(\overline{Y}) \oplus \bigoplus_{i,j} \mathbb{Z}[E_j^i]$ with $\text{Pic}(\overline{Y})^\perp = \bigoplus_{i,j} \mathbb{Z}[E_j^i]$. We choose not to work such a decomposition for presentational reasons.

We will describe the tropicalisations of $\alpha_{\mathbf{t}}$ and $\alpha_{\mathbf{nt}}$ individually to begin with. We make the following definition which will be used to record how elements of D^\perp arise from elements of $\text{Pic}(\overline{Y})$.

Definition 4.18. A *contact order* is a tuple $(\check{\lambda}_i)_{i \in I^h}$ with $\check{\lambda}_i \in \check{\Lambda}_{\rho_i}$.

Example 4.19. Let (Y, D) be a toric Looijenga pair and β a curve class on Y . Then the contact order associated to β is $((\beta \cdot D_i)\check{\mathbf{e}}_i)_{i \in I^h}$ where $\check{\mathbf{e}}_i \in \check{\Lambda}_{\rho_i}$ is the primitive unit vector corresponding to 1.

We also have the converse: given an element $\alpha_{\mathbf{t}} = \sum_i \beta_i D_i \in \text{Pic}(\overline{Y})$, the contact order completely determines it [FS97, Theorem 2.1]. We thus define the tropicalisation $\text{Trop}(\alpha_{\mathbf{t}})$ to the subset of $\overline{\text{Sk}}(U)$ consisting of the closure of the rays ρ_i whenever $\alpha_{\mathbf{t}} \cdot D_i \neq 0$ equipped with the weight $(\alpha_{\mathbf{t}} \cdot D_i)\check{\mathbf{e}}_i$. By [Ful93, §3], the divisor $\alpha_{\mathbf{t}}$ defines a continuous piecewise linear function $\phi_{\alpha_{\mathbf{t}}}$ on the support of the fan $\overline{\Sigma}$ of \overline{Y} given by $\phi_{\alpha_{\mathbf{t}}}(\mathbf{v}_i) = -\alpha_{\mathbf{t}} \cdot D_i$ and extended linearly, where \mathbf{v}_i is the primitive integral generator of the ray in $\overline{\Sigma}$ corresponding to D_i . Since $\phi_{\alpha_{\mathbf{t}}}$ is defined at the origin, we must have $\sum_i (-\alpha_{\mathbf{t}} \cdot D_i)\mathbf{v}_i = 0$. In particular, the tropicalisation is balanced at O and thus defines an infinite tropical spoke.

We next describe the tropicalisation of a non-toric exceptional curve E_j^i on Y . Then the image of $(E_j^i)^{\text{an}}$ under the non-archimedean SYZ fibration $\rho_{(Y,D)}$ is the ray emanating from v_j^i along ρ_i ; we denote this ray by r_j^i . Then the tropicalisation of γE with $\gamma \in \mathbb{Z}$ is the ray r_j^i equipped with the weight $\gamma \in \check{\Lambda}_{\rho_i}$. We record the following observation:

Lemma 4.20. Let $\alpha \in D^\perp$ and write $\alpha = \alpha_{\mathbf{t}} + \alpha_{\mathbf{nt}} = \sum_i \beta_i D_i + \sum_{i,j} \gamma_j^i E_j^i$ with $\alpha_{\mathbf{t}} \neq 0$. If there exists a j such that $\gamma_j^i \neq 0$, then $\alpha_{\mathbf{t}} \cdot D_i \neq 0$. If j is unique, then $\alpha_{\mathbf{t}} \cdot D_i = \gamma_j^i$.

Proof. Since $(\alpha_{\mathbf{t}} + \alpha_{\mathbf{nt}}) \cdot D_i = 0$ and $\alpha_{\mathbf{nt}} \cdot D_i \neq 0$, we have $\alpha_{\mathbf{t}} \cdot D_i \neq 0$. \square

Given $\alpha \in D^\perp$, we can always write α as a sum of elements of $D^\perp \cap \bigoplus_{r,s} \mathbb{Z}[E_s^r]$ and elements of the form

$$\sum_i \beta_i D_i + \sum_{i,j} \gamma_j^i E_j^i$$

such that for each i there exists a unique j such that $\gamma_j^i \neq 0$. Indeed, write $\alpha = \alpha_{\mathbf{t}} + \alpha_{\mathbf{nt}}$ and suppose there exist j and j' such that $\gamma_j^i, \gamma_{j'}^i \neq 0$. Then $\epsilon = E_j^i - E_{j'}^i \in D^\perp$ and we can run the same argument on $\alpha \pm \epsilon$ until we have the desired form. This leads to the following definition.

I'm unsure about this definition. Is this saying for each i , there exists a unique exceptional curve, E intersecting D_i such that α contains a non-zero contribution from E ?

Definition 4.21. We call $\alpha \in D^\perp$ *primitive* if for each i , there exists a unique j such that $\gamma_j^i \neq 0$.

Lemma 4.22. Denote the subgroup of D^\perp of classes with $\alpha_t = 0$ by D_E^\perp . Then we have

$$D^\perp = D_s^\perp \oplus D_E^\perp$$

Proof. Let $\alpha \in D^\perp$. Then we can write $\alpha = \alpha_t + \alpha_{nt}$. □

We can now fully describe tropicalisation of $\alpha = \alpha_t + \alpha_{nt}$. There are two cases to consider given by whether α_t is zero or non-zero.

In the case $\alpha_t = 0$, then for each i , we have $\sum_j \gamma_j^i = 0$. In particular, there is a set of pairs $J = \{(j_1, j_2) : \gamma_{j_1}^i, \gamma_{j_2}^i \neq 0\}$ such that we have

$$\alpha_{nt} = \sum_i \sum_{(j_1, j_2) \in J} (E_{j_2}^i - E_{j_1}^i).$$

Then the tropicalisation of α will be the sum of the realised tropical wings on each ray ρ_i connecting the singularities $v_{j_1}^i$ and $v_{j_2}^i$ with weight given by $\text{sign}(j_2 - j_1) \in \check{\Lambda}_{\rho_i}$.

In the case $\alpha_t \neq 0$, we proceed as follows. We have seen that the tropicalisation of α_t is an infinite tropical spoke. By the discussion above, we can assume that α is primitive and write $\alpha = \sum_i \beta_i D_i + \sum_j \gamma_j^i E_j^i$. The tropicalisation of α_{nt} consists of the rays r_j^i emanating from the corresponding singularities in $\text{Sk}(U)$ with weight $\gamma_j^i \in \check{\Lambda}_{\rho_i} \cong \mathbb{Z}$. We define $\text{Trop}(\alpha)$ to be the finite tropical spoke given by removing the rays r_j^i from $\text{Trop}(\alpha_t)$. The weights are the same as those of the tropicalisation of α_t by Lemma 4.20 and $\text{Trop}(\alpha)$ is indeed a realised finite tropical spoke by the same argument that showed $\text{Trop}(\alpha_t)$ is a finite tropical spoke. This completely describes Trop . The following example captures all aspects of the construction of Trop .

Example 4.23. Consider \mathbb{P}^2 blown up at 4 points in the configuration in Figure 11. Then D^\perp is a rank 2 lattice generated by $\alpha = [L] - [E_1] - [E_2] - [E_3]$ and $\beta = [E_1] - [E_4]$ where E_1 and E_4 are the orange and blue exceptional curves respectively. The tropicalisation of α is a tropical spoke (in yellow) and β is a tropical wing (in purple) as drawn on the right hand side.

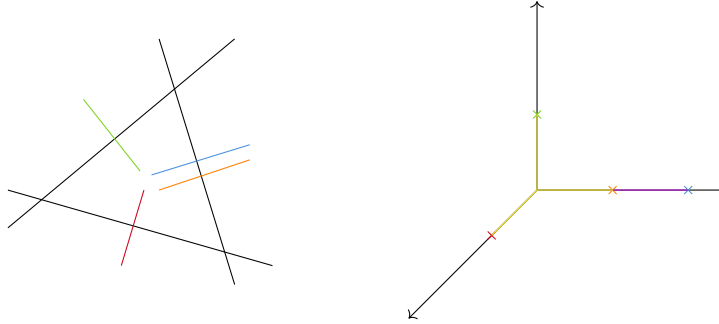


FIGURE 11. Tropicalisation of D^\perp for non-toric dP5

Remark 4.24. Any algebraic cycle on Y can be expressed as a sum of curve classes $\sum_j \beta_j [C_j]$ where C_j does not pass through any 0-dimensional stratum in \mathcal{Y} . When an element $\alpha \in D^\perp$ can be represented by an irreducible curve C_α in U which does not pass through any 0-dimensional stratum in \mathcal{Y} , the topological realisation of a tropical spoke/wing agrees with $\rho_{(\mathcal{Y}, \mathcal{D})}(C_\alpha^{\text{an}})$ under the non-archimedean SYZ fibration $\rho_{(\mathcal{Y}, \mathcal{D})}$.

For example, α and β in Example 4.23 can be realised by irreducible (-2) -curves C_α and C_β in U respectively. Then $\rho_{(Y,D)}(C_\alpha^{\text{an}})$ and $\rho_{(Y,D)}(C_\beta^{\text{an}})$ coincide with the tropicalisation depicted in Figure 11.

Let $D_{\mathbf{t}}^\perp$ denote the subgroup of D^\perp generated by the primitive elements of D^\perp . In particular, if $\alpha \in D_{\mathbf{t}}^\perp$ then $\alpha_{\mathbf{t}} \neq 0$. Let D_{nt}^\perp denote the subgroup of D^\perp with $\alpha_{\mathbf{t}} = 0$. We then have the decomposition $D^\perp = D_{\mathbf{t}}^\perp \oplus D_{\text{nt}}^\perp$. This decomposition is compatible with $\underline{\text{TropSpoke}} \oplus \underline{\text{TropWing}}$:

Proposition 4.25. The homomorphism $\text{Trop} : D_{\mathbf{t}}^\perp \rightarrow \underline{\text{TropSpoke}}$ is an isomorphism.

Proof. From the construction of Trop , it is clear that Trop is a group homomorphism. Since $\alpha_{\mathbf{t}}$ is completely determined by its contact order, we have that Trop is injective. We now construct an inverse homomorphism; to do this, we will use the equivalence between Cartier divisors and piecewise linear functions on $\bar{\Sigma}$ as we did above. Let τ be a tropical spoke. We can naturally extend τ to an infinite tropical spoke τ_∞ with the weights induced by τ . This defines α_{nt} . We are left to construct a cycle $\alpha_{\mathbf{t}} \in D_{\mathbf{t}}^\perp$ with contact order given by the tuple of weights on τ_∞ . More precisely, suppose τ_∞ consists of the rays ρ_i with weight w_i . By the balancing condition, we have $\sum_i w_i \tilde{\mathbf{e}}_i = 0 \in \check{\Lambda}_0$. Consider the function $\phi_{\alpha_{\mathbf{t}}} : |\bar{\Sigma}| \rightarrow \mathbb{R}$ defined above with $\phi_D(\mathbf{v}_i) = -w_i \in \mathbb{Z}$. This function specifies a divisor on \bar{Y} with the contact order $(\alpha_{\mathbf{t}} \cdot D_i)_i$ by [Ful93, §3]. We can pull this divisor back to Y and this is precisely $\alpha_{\mathbf{t}}$. This describes the inverse map to Trop . \square

We have now identified the cycles in D^\perp with tropical objects on the base of the non-archimedean SYZ fibration. Our next step is to compute the rank of D^\perp in terms of these tropical objects. By the classification of surfaces, we have the following elementary counting argument.

Proposition 4.26. Assume there are k_i singularities along the ray ρ_i in $\text{Sk}(U)$ and exactly $k+2$ rays have singularities. Then

$$(5) \quad \begin{aligned} \text{rank}(D^\perp) &= \# \text{ simple tropical wings} + \# \text{ simple tropical spokes.} \\ &= \begin{cases} \sum_{i: k_i \neq 0} (k_i - 1) + k = \sum_i k_i - 2 & \text{if } k \geq 1 \\ \sum_{i: k_i \neq 0} (k_i - 1) + T_2 = \sum_i k_i - 2 + T_2 & \text{if } k = 0 \end{cases} \end{aligned}$$

where $T_2 \in \{0, 1\}$ is equal to the number of realised simple tropical spokes consisting of exactly two vertices.

Proof. We prove this result by showing equality of the right hand side of (5) with the expression given in Lemma 4.4. We will prove the result by induction on the number of non-toric exceptional curves, or equivalently the number of singularities in $\text{Sk}(U)$. The toric model (\bar{Y}, \bar{D}) is either \mathbb{P}^2 , \mathbb{F}_n or a toric blowup of one of these. We analyse each case separately. To begin with, if (Y, D) is toric then both sides agree. Thus we turn our attention to the non-toric case. Let $\rho = \text{rank}(\text{Pic}(Y)) = \sum_i k_i + \text{rank}(\text{Pic}(\bar{Y}))$ and recall $s = \text{rank}(\bigoplus_i \mathbb{Z}[D_i] \rightarrow \text{Pic}(Y))$.

(i) $(\bar{Y}, \bar{D}) = (\mathbb{P}^2, \bar{D}_1 + \bar{D}_2 + \bar{D}_3)$: in this case, by Lemma 4.4 we have

$$\text{rank}(D^\perp) = \rho - 3 + s = \sum_i k_i + s - 2,$$

where $s = 0$ if two rays have singularities or all three rays have singularities, $s = 1$ if all the singularities are on one ray and $s = 2$ if (Y, D) is toric. If exactly two rays have singularities, without loss of generality, we can assume $k_3 = 0$. Then the equality (5) holds as there are no simple tropical spokes. If all three rays have singularities, then there is

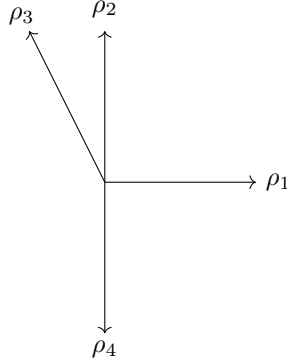
exactly one simple tropical spoke and thus the desired equality also holds. In the $s = 1$ case, we can assume $k_2 = k_3 = 0$ and the equality holds.

(ii) $(\bar{Y}, \bar{D}) = (\mathbb{F}_n, \bar{D}_1 + \bar{D}_2 + \bar{D}_3 + \bar{D}_4)$ with $n \geq 1$: in this case, we have

$$\text{rank}(D^\perp) = \rho - 4 + s = \sum_i k_i + s - 2.$$

We adopt the following notation. Let $\text{Pic}(\bar{Y}) = \langle C, F \rangle$ where C is the unique irreducible curve of self-intersection $-n$ and F is a fibre. Then we write

$$\bar{D}_1 = F, \bar{D}_2 = nF + C, \bar{D}_3 = F \text{ and } \bar{D}_4 = C.$$



As in the previous case, if all the singularities are on one ray, then $s = 1$ and we can assume $k_2 = k_3 = k_4 = 0$. This gives the desired equality. If exactly two rays have singularities, there are two cases to consider. If the two rays which have singularities correspond to \bar{D}_2 and \bar{D}_4 *i.e.* lie on a line of the fan of (\bar{Y}, \bar{D}) , then $s = 1$. If they do not lie on a line, then $s = 0$. If exactly three rays have singularities, then $s = 0$ and there are three cases to consider. If the rays which have singularities correspond to \bar{D}_1, \bar{D}_2 and \bar{D}_3 or \bar{D}_1, \bar{D}_3 and \bar{D}_4 , then there is exactly one simple tropical spoke. In the other two cases, there are no simple tropical spokes involving all rays; there is still exactly one simple tropical spoke coming from \bar{D}_2 and \bar{D}_4 . If all four rays have singularities, then $s = 0$ and there are exactly two simple tropical spokes. In summary,

$$\text{rank}(D^\perp) = \begin{cases} k_1 - 1 & \text{if } k_2 = k_3 = k_4 = 0 \\ k_2 + k_4 - 1 & \text{if } k_1 = k_3 = 0, k_2, k_4 \neq 0 \\ k_i + k_j - 2 & \text{if } \{i, j\} \neq \{2, 4\} \\ k_1 + k_2 + k_3 - 2 & \text{if } k_4 = 0 \\ k_1 + k_3 + k_4 - 2 & \text{if } k_2 = 0 \\ k_2 + k_4 + k_l - 2 & \text{if } k_l \neq 0 \\ k_1 + k_2 + k_3 + k_4 - 2 & \text{if } k_i \neq 0 \forall i \end{cases}$$

which agrees with the right hand side of (5).

(iii) Suppose that (\bar{Y}, \bar{D}) is a single toric blowup of \mathbb{P}^2 or \mathbb{F}_n . Denote the boundary exceptional curve by E . Assume that there is a non-toric blowup on E , else we can reduce to the

previous cases. We have

$$\text{rank}(D^\perp) = \rho - \ell(D) + s = \sum_i k_i + s - 2.$$

If the only singularities are on the ray corresponding to E , then $s = 1$ and thus $\text{rank}(D^\perp) = k_i - 1$ for some i . This gives us the equality since there are no simple tropical spokes. If two rays have singularities, then there are two cases. The first is if the singularities all lie on a line of the fan of (\bar{Y}, \bar{D}) . In this case, $s = 1$ and $\text{rank}(D^\perp) = k_i + k_j - 1$ which agrees with the right hand side of (5) since there is exactly one simple tropical spoke. If the singularities do not lie on a line, then $s = 0$ and $\text{rank}(D^\perp) = k_i + k_j - 2$. By Lemma 4.16, if $2 + k$ rays have singularities with $k \geq 1$, then $s = 0$ and there are exactly k simple tropical spokes. The right hand side of (5) is then $\sum_{i:k_i \neq 0} (k_i - 1) + k = \sum_i k_i - 2 - k + k = \sum_i k_i - 2 = \text{rank}(D^\perp)$. We can then induct on the number of toric blowups and repeat the argument above. \square

Corollary 4.27. The map $\text{Trop} : D^\perp \rightarrow \underline{\text{TropSpoke}} \oplus \underline{\text{TropWing}}$ is an isomorphism of groups.

Proof. By Proposition 4.26, it remains to prove that Trop is surjective. Proposition 4.25 exhibits surjectivity onto $\underline{\text{TropSpoke}}$ and thus we are left to prove Trop is surjective onto $\underline{\text{TropWing}}$; we can check the claim for simple tropical wings. Given a positively oriented simple tropical wing f associated to the edge $\tau_{j,j+1}^i$, we have $\text{Trop}(E_{j+1}^i - E_j^i) = f$. This completes the proof. \square

4.3. Tropical 1-cycles.

Definition 4.28. A tropical 1-cycle is a twisted singular 1-cycle on $\text{Sk}(U) \setminus \Gamma$ with coefficients in the sheaf of integral cotangent vectors $\check{\Lambda}$ on $\text{Sk}(U) \setminus \Gamma$.

More concretely, we can describe a tropical 1-cycle as an oriented graph G with a map $f : G \rightarrow \text{Sk}(U) \setminus \Gamma$ and a section $\check{\xi}_e \in (f_e)^* \check{\Lambda}$ for each edge $e \subset G$. Furthermore, at each vertex v of G , we have the balancing condition

$$\sum_{e \ni v} \epsilon_e \check{\xi}_e = 0$$

where $\epsilon_e = 1$ if e is an incoming edge and $\epsilon_e = -1$ if e is an outgoing edge. We will depict a tropical cycle (locally) as in Figure 12, with each section $\check{\xi}_e$ drawn by taking the corresponding dual tangent vector $\xi_e \in \Lambda$.

Lemma 4.29 ([LZ23]). If a tropical cycle goes around a focus-focus singularity on ρ i.e. given by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with respect to a suitable oriented basis where $\check{\Lambda}_\rho$ is the invariant direction, as shown in Figure 12, then the edge emanating from the singularity carries a cotangent vector ζ parallel to $\check{\Lambda}_\rho$. Moreover, ζ is primitive if and only if the cycle carries a primitive generator of $\check{\Lambda}/\check{\Lambda}_\rho$ when it starts to surround the singularity.

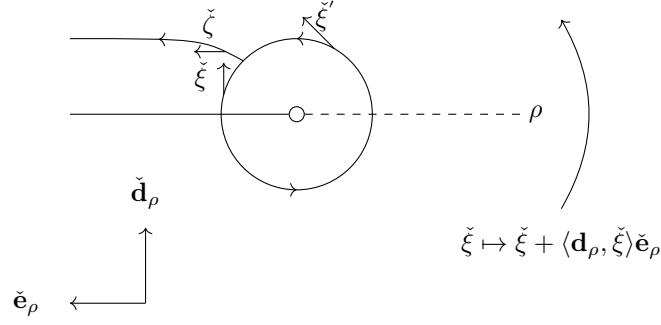


FIGURE 12. Tropical cycle surrounding a focus-focus singularity.

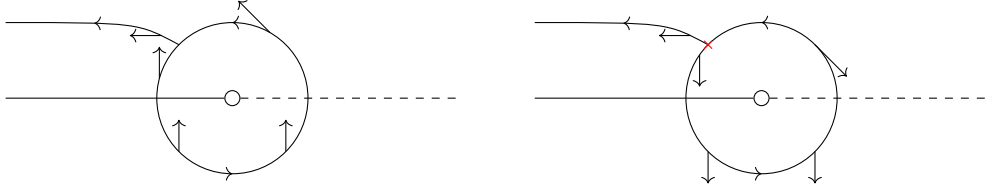
Proof. Let \mathbf{e}_ρ be a generator of Λ_ρ which points away from the origin O and $\check{\mathbf{e}}_\rho \in \check{\Lambda}_\rho$ the dual cotangent vector such that $\langle \mathbf{e}_\rho, \check{\mathbf{e}}_\rho \rangle = 1$. Furthermore, let $\check{\Lambda}_\rho^\perp = \langle \mathbf{d}_\rho \rangle$ and $\check{\mathbf{d}}_\rho$ be the dual cotangent vector.

As depicted in Figure 12, let ξ be the cotangent vector the tropical 1-cycle carries as it starts to go around the focus-focus singularity and ξ' the vector after applying the monodromy transformation to ξ . By the balancing condition, we must have

$$\xi + \check{\zeta} = \xi' = \xi + \langle \mathbf{d}_\rho, \xi \rangle \check{\mathbf{e}}_\rho$$

which implies $\check{\zeta} = \langle \mathbf{d}_\rho, \xi \rangle \check{\mathbf{e}}_\rho$. Thus $\check{\zeta}$ is parallel to $\check{\Lambda}_\rho$ and it is a primitive generator of $\check{\Lambda}_\rho$ if and only if ξ is a primitive generator of $\check{\Lambda}/\check{\Lambda}_\rho$. \square

Remark 4.30. Lemma 4.29 gives an explicit local description of the tropical cycles of interest to us. In particular, we will give a basis of $H_1(\text{Sk}(U), \iota_* \check{\Lambda})$ that looks locally like Figure 12. Note that the decoration of the loop around the focus-focus singularity is unique once we fix the orientation of the edge emanating from the singularity, its decoration and the orientation of the loop. This is due to the balancing condition - for example, the figure below on the left satisfies the balancing condition but the one on the right is not balanced at the red cross.



We will discuss the case when there are multiple edges emanating from the loop in Remark 4.33.

We will now compute the rank of $H_1(\text{Sk}(U), \iota_* \check{\Lambda})$. Here it is essential that the essential skeleton $\text{Sk}(U)$ arises from the model $(\mathcal{Y}, \mathcal{D})$ from Proposition 3.34 so that the rank of $H_1(\text{Sk}(U), \iota_* \check{\Lambda})$ agrees with the rank of D^\perp .

Lemma 4.31. We have the following equality:

$$\text{rank}(H_1(\text{Sk}(U)^{\text{sm}}, \check{\Lambda})) = 2 \left(\sum_i k_i \right) - 2 + \text{rank}(H_0(\text{Sk}(U)^{\text{sm}}, \check{\Lambda})).$$

Proof. By [Dim04, Proposition 2.5.4], we have $\chi(\mathrm{Sk}(U)^{\mathrm{sm}}, \check{\Lambda}) = \mathrm{rank}(\check{\Lambda}) \cdot \chi(\mathrm{Sk}(U)^{\mathrm{sm}})$. Rearranging and using the fact that $\mathrm{Sk}(U)^{\mathrm{sm}}$ is homeomorphic to \mathbb{R}^2 punctured at $\sum_i k_i$ distinct points, we have

$$\mathrm{rank}(H_1(\mathrm{Sk}(U)^{\mathrm{sm}}, \check{\Lambda})) = 2 \left(\sum_i k_i \right) - 2 + \mathrm{rank}(H_0(\mathrm{Sk}(U)^{\mathrm{sm}}, \check{\Lambda}))$$

as required. \square

Consider the following short exact sequence of chain complexes

$$0 \rightarrow C_\bullet(\Gamma, \iota_* \check{\Lambda}) \rightarrow C_\bullet(\mathrm{Sk}(U), \iota_* \check{\Lambda}) \rightarrow C_\bullet(\mathrm{Sk}(U), \Gamma; \iota_* \check{\Lambda}) \rightarrow 0$$

where $C_n(\mathrm{Sk}(U), \Gamma; \iota_* \check{\Lambda}) = C_n(\mathrm{Sk}(U), \iota_* \check{\Lambda}) / C_n(\Gamma, \iota_* \check{\Lambda})$. Taking the long exact sequence in homology gives the following exact sequence:

$$(6) \quad 0 \rightarrow H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda}) \rightarrow H_1(\mathrm{Sk}(U), \Gamma; \iota_* \check{\Lambda}) \rightarrow H_0(\Gamma, \iota_* \check{\Lambda}) \rightarrow H_0(\mathrm{Sk}(U), \iota_* \check{\Lambda}) \rightarrow 0.$$

Suppose that $\mathrm{Sk}(U)$ has exactly $k + 2$ rays which have a singularity. Then by direct computation, we have that

$$\mathrm{rank}(H_1(\mathrm{Sk}(U), \Gamma; \iota_* \check{\Lambda})) = \begin{cases} \sum_i k_i + \sum_{i: k_i \neq 0} (k_i - 1) + k, & \text{if } k \geq 1 \\ \sum_i k_i + \sum_{i: k_i \neq 0} (k_i - 1) + T_2 & \text{if } k = 0 \end{cases}$$

where $T_2 \in \{0, 1\}$ is equal to the number of realised simple tropical spokes consisting of exactly two vertices. Rearranging gives

$$\begin{aligned} \mathrm{rank}(H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})) &= \mathrm{rank}(H_1(\mathrm{Sk}(U), \Gamma; \iota_* \check{\Lambda})) - \sum_i k_i \\ &\quad - \mathrm{rank}(\ker(H_0(\Gamma, \iota_* \check{\Lambda}) \rightarrow H_0(\mathrm{Sk}(U), \iota_* \check{\Lambda}))). \end{aligned}$$

There are two cases to consider now: (i) if the singularities all lie on a line in the fan $\bar{\Sigma}$ of the toric model \bar{Y} , then $\mathrm{rank}(H_0(\mathrm{Sk}(U), \iota_* \check{\Lambda})) = 1$ *i.e.* there is a global cotangent vector on $\mathrm{Sk}(U)$; (ii) if all the singularities do not lie on a line, then $\mathrm{rank}(H_0(\mathrm{Sk}(U), \iota_* \check{\Lambda})) = 0$. In both cases, we have

$$\mathrm{rank}(H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})) = \begin{cases} \sum_{i: k_i \neq 0} (k_i - 1) + k & \text{if } k \geq 1 \\ \sum_{i: k_i \neq 0} (k_i - 1) + T_2 & \text{if } k = 0 \end{cases}$$

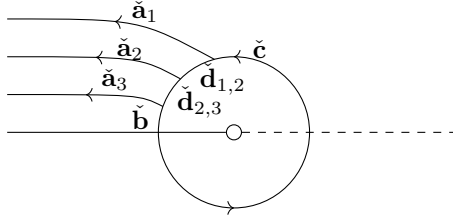
which agrees with the rank of D^\perp . This proves:

Lemma 4.32. We have the equality

$$\mathrm{rank}(D^\perp) = \mathrm{rank}(H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})).$$

Observe that the map $H_1(\mathrm{Sk}(U)^{\mathrm{sm}}, \check{\Lambda}) \rightarrow H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})_{\mathrm{tf}}$ to the torsion-free component of $H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})$ is surjective and the kernel has rank $\sum_i k_i$. A generating set of the kernel is given by loops around each singularity decorated with the invariant primitive cotangent vector.

Remark 4.33. Instead of having a single emanating edge from a loop around a singular vertex as depicted in Figure 12, consider the case where there are $k \geq 2$ emanating edges pointing away from the vertex. The figure below illustrates the case $k = 3$. Let each edge be decorated with the cotangent vector $\check{\mathbf{a}}_i$ with $\check{\mathbf{c}}$ the image of $\check{\mathbf{b}}$ under the monodromy transformation.



From the exact sequence (6), we have the morphism

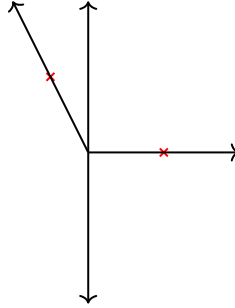
$$H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda}) \rightarrow H_1(\mathrm{Sk}(U), \Gamma; \iota_* \check{\Lambda}).$$

The image of such a tropical cycle under this morphism will have k edges emanating from the singular vertex, and each will be decorated with the invariant cotangent vector with the relation $\sum_i \check{\mathbf{a}}_i = 0$. We also have the balancing conditions

$$\begin{aligned} \check{\mathbf{a}}_1 + \check{\mathbf{c}} &= \check{\mathbf{d}}_{1,2} \\ \check{\mathbf{a}}_i + \check{\mathbf{d}}_{i-1,i} &= \check{\mathbf{d}}_{i,i+1} \text{ for } 1 < i < k \\ \check{\mathbf{a}}_k + \check{\mathbf{d}}_{k-1,k} &= \check{\mathbf{b}}. \end{aligned}$$

This implies $\mathbf{b} = \mathbf{c}$ and in particular \mathbf{b} is parallel to the invariant cotangent direction. In particular, a tropical 1-cycle can be locally depicted as having one emanating edge.

Remark 4.34. As observed in [Rud21], $H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})$ may have torsion. For example, consider the Looijenga pair (Y, D) with toric model given by \mathbb{F}_2 and non-toric blowups along both fibres, whose essential skeleton is depicted below. Since the invariant cotangent directions do not span $\check{\Lambda}_O$, a simple Čech homology computation shows that $H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda}) \cong \mathbb{Z}/2\mathbb{Z}$. The torsion elements of $H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})$ have no importance in our theory and we will only work with the torsion-free part. In the work of [RS20], the construction of the mirror family depends on a choice of splitting of $H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda}) \rightarrow H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})_{\mathrm{tf}}$ by the torsion subgroup $H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})_{\mathrm{tors}}$.



Theorem 4.35. (Tropical correspondence) There is an isomorphism of groups

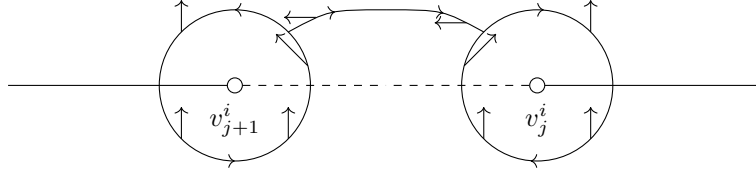
$$T : D^\perp \longrightarrow H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})_{\mathrm{tf}}.$$

Proof. By Corollary 4.27, it is sufficient to prove there is an isomorphism

$$\underline{\mathrm{TropSpoke}} \oplus \underline{\mathrm{TropWing}} \longrightarrow H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})_{\mathrm{tf}}.$$

Let f be a realised finite tropical spoke which consists of the vertices $v_{j_i}^i$ and edges $\tau_{j_i}^i$ with $v_{j_i}^i \in \rho_i$. For each vertex $v_{j_i}^i$ of f , we consider a positively oriented loop around the vertex such that the edge emanating away from the vertex with the orientation towards the origin is equipped with the vector $f(v_{j_i}^i)\check{\mathbf{e}}_i$ as depicted in Figure 12. By Lemma 4.29 and Remark 4.30, the decoration on the loop is then completely determined. The emanating edges meet at O at which the balancing condition $\sum_i f(v_{j_i}^i)\check{\mathbf{e}}_i = 0$ is satisfied. This describes a tropical cycle $\gamma_f \in H_1(\mathrm{Sk}(U), \iota_*\check{\Lambda})$ associated to the tropical spoke f .

If f is a positively oriented tropical wing corresponding to the edge $\tau_{j,j+1}^i$, consider a positively oriented loop around the vertex v_{j+1}^i and a negatively oriented loop around the vertex v_j^i . The emanating edges from each loop can be glued together and equipped with the vector $1 \in \check{\Lambda}_{\rho_i}$, which preserves the orientation and decoration and, by Remark 4.30 again, determines the decoration of the loops as depicted below:



This describes a tropical cycle $\gamma_f \in H_1(\mathrm{Sk}(U), \iota_*\check{\Lambda})_{\mathrm{tf}}$. By construction, the mapping $f \mapsto \gamma_f$ is a group homomorphism. Given a tropical cycle γ , there is a representative of γ such that each edge emanating from a loop around a singularity $v_j^i \in \rho_i$ is contained in the ray $\rho_i \subseteq \mathrm{Sk}(U)$. By contracting the loop around each vertex, the resulting graph will be equal to a sum of realised tropical spokes/wings. This proves surjectivity of the map and thus by the rank computation above, we are done. \square

Example 4.36. On the left of Figure 13, we have a tropical cycle for the Looijenga pair arising from performing a single non-toric blowup on each boundary component of \mathbb{P}^2 . Under our correspondence, this tropical cycle corresponds to $[L] - [E_1] - [E_2] - [E_3]$ where E_i are the exceptional curves and L is a representative of the pullback of the line class on \mathbb{P}^2 .

On the right of the figure, we have a tropical cycle for the Looijenga pair with toric model \mathbb{F}_1 by performing a non-toric blowup along one of the fibres and the other two components. The tropical cycle corresponds to $-[C] + [E_2] - [E_1] - [E_3]$ where C is the pullback of the (-1) -curve on \mathbb{F}_1 and E_2 is the exceptional curve with centre on a fibre of \mathbb{F}_1 .

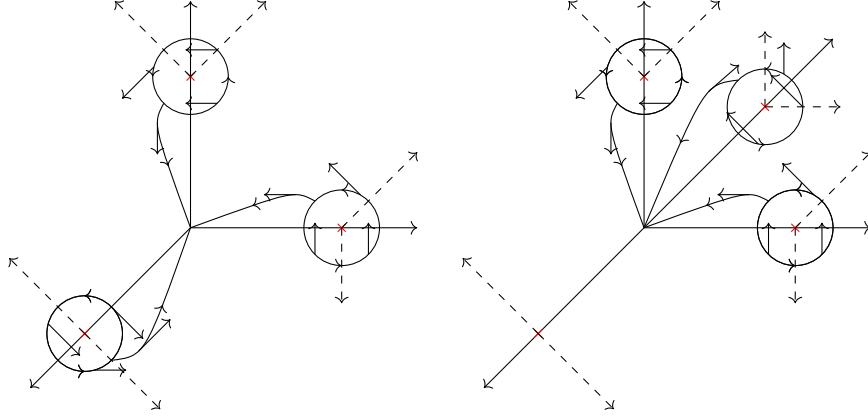


FIGURE 13. Tropical cycles corresponding to tropical spokes.

Definition 4.37. Let $\gamma \in H_1(\mathrm{Sk}(U)^{\mathrm{sm}}, \check{\Lambda})$. We say γ is

- (1) an *invariant cycle* if it is given by a loop around a singularity decorated with an invariant cotangent vector;
- (2) an *exceptional tropical cycle* if under the tropical correspondence 4.35, γ corresponds to a simple tropical wing;
- (3) a *spoked tropical cycle* if under the tropical correspondence 4.35, γ corresponds to a primitive tropical spoke.

By the rank computation above, we see that these elements generate $H_1(\mathrm{Sk}(U)^{\mathrm{sm}}, \check{\Lambda})$ and that the exceptional and spoked tropical cycles generate $H_1(\mathrm{Sk}(U), \iota_*\check{\Lambda})$.

Remark 4.38. Given $L \in D^\perp$, if $L = \alpha_\mathfrak{t} + \sum_{i,j} \gamma_j^i E_j^i$ corresponds to a primitive tropical spoke f , then the coefficient γ_j^i is equal to $f(v_j^i)$ by construction. If $L = E_j^i - E_k^i$ with $k = j \pm 1$, then L corresponds to the simple tropical wing f and $f(\tau_{j,k}^i)$ is equal to the sign of $j - k$.

5. K -AFFINE STRUCTURES

In this section, we will present the general theory of K -affine structures from [KS06], a concept that resides in between the worlds of tropical and analytic geometry. We will see that a K -affine structure is an enhancement of an integral affine structure. In §5.1, we introduce the necessary tools to compute an explicit K -affine structure (5.15) that will be used in the definition of a non-archimedean period. In §5.2, we will describe how to compute the K -affine structure using Čech cohomology. The first application of K -affine structures will be to show how they completely determine a maximally degenerate abelian variety over K . We will do this in §5.3 for the more general class of toric abeloid K -varieties which are the analogues of complex tori for non-archimedean fields. We then explicitly describe the K -affine structure that will be used in the period computation for a log Calabi-Yau surface using Čech cohomology. In §5.4, we end with a computation of the monodromy representation (5.35) of the K -affine structure from Theorem 5.15 for a log Calabi-Yau surface.

5.1. Construction.

Definition 5.1. Let B be an integral affine manifold. Denote by $\text{Aff}_{\mathbb{Z}}$ the sheaf of integral affine functions on B ; this fits into the short exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \text{Aff}_{\mathbb{Z}} \longrightarrow \check{\Lambda} \longrightarrow 0.$$

A K -affine structure on B is an abelian sheaf \mathcal{F} on B which fits into the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K^{\times} & \longrightarrow & \mathcal{F} & \longrightarrow & \check{\Lambda} \longrightarrow 0 \\ & & \downarrow v_K & & \downarrow & & \downarrow \text{id} \\ 0 & \longrightarrow & \mathbb{R} & \longrightarrow & \text{Aff}_{\mathbb{Z}} & \longrightarrow & \check{\Lambda} \longrightarrow 0 \end{array}$$

where the rows are short exact sequences.

Example 5.2. Let $B = \mathbb{R}^n$ with the standard integral affine structure given by the embedding $\mathbb{Z}^n \subseteq \mathbb{R}^n$. Then any K -affine structure on B is isomorphic to $\check{\Lambda} \oplus K^{\times}$ since the short exact sequence above always splits.

Remark 5.3. A K -affine structure on B is a $\text{GL}_n(\mathbb{Z}) \times (K^{\times})^n$ -torsor on B such that applying v_K^n to $(K^{\times})^n$ gives the $\text{GL}_n(\mathbb{Z}) \times \mathbb{R}^n$ -torsor associated to the integral affine structure on B .

Definition 5.4. To a K -affine structure \mathcal{F} , we can associate a monodromy representation

$$m : \pi_1(B, b) \longrightarrow \text{GL}_n(\mathbb{Z}) \times (K^{\times})^n$$

for all $b \in B$ by composing the transition functions of Aff_K as we go around the loop. Post-composing with v_K^n then recovers the monodromy representation associated to the integral affine structure $\text{Aff}_{\mathbb{Z}}$.

As noted in §2.7, an affinoid torus fibration induces an integral affine structure on the target. We review this construction below. It is sufficient to consider the case $X = \mathbb{G}_m^n$ with $B = \mathbb{R}^n$. Then $\text{trop} : X^{\text{an}} \rightarrow B$ is an affinoid torus fibration trivially. We first describe the case $X = \mathbb{G}_m^n$ and $B = \mathbb{R}^n$ with the affinoid torus fibration $\text{trop} : X^{\text{an}} \rightarrow B$. Let $V \subset \mathbb{R}^n$ be a connected open set. In coordinates (z_1, \dots, z_n) on $\text{trop}^{-1}(V) \subset \mathbb{G}_m^{n, \text{an}}$, every invertible analytic function $\varphi \in \mathcal{O}_{\mathbb{G}_m^{n, \text{an}}}^{\times}(\text{trop}^{-1}(V))$ can be written as a Laurent series

$$\varphi = \sum_{I \in \mathbb{Z}^n} c_I z^I, c_I \in K$$

such that for every $\underline{x} = (x_1, \dots, x_n) \in V$, $-\log |c_I| + \sum_{i=1}^n x_i I_i \rightarrow +\infty$ as $|I| \rightarrow \infty$.

Definition 5.5. Let $\varphi \in \mathcal{O}_{\mathbb{G}_m^{n, \text{an}}}^{\times}(\text{trop}^{-1}(V))$ and $x \in V$. By the maximum modulus principle, $|\varphi|$ is constant on the fibres of trop . Let $|\varphi(x)|$ denote the value of $|\varphi|$ on the fibre $\text{trop}^{-1}(x)$ and define

$$\text{Val}(\varphi)(x) := -\log |\varphi(x)| = \inf_I \left(-\log |c_I| + \sum_i x_i I_i \right).$$

Lemma 5.6. The map $\text{Val} : \text{trop}_*(\mathcal{O}_{(\mathbb{G}_m^n)^{\text{an}}}^{\times}) \rightarrow \text{Aff}_{\mathbb{Z}}$ of sheaves on \mathbb{R}^n satisfies the following properties:

- (i) Let $a \in K^{\times}$. Then $\text{Val}(a) = -\log |a|$.

- (ii) Let $V \subset \mathbb{R}^n$ be a connected open set and $\varphi \in \mathcal{O}_{(\mathbb{G}_m^n)_{\text{an}}}^\times(\text{trop}^{-1}(V))$. There exists a unique index I , such that for all $x \in V$,

$$\text{Val}(\varphi)(x) = -\log |c_I| + \sum_i x_i I_i.$$

- (iii) Given $\varphi_1, \varphi_2 \in \mathcal{O}_{(\mathbb{G}_m^n)_{\text{an}}}^\times(\text{trop}^{-1}(V))$, we have

$$\text{Val}(\varphi_1 \varphi_2) = \text{Val}(\varphi_1) + \text{Val}(\varphi_2).$$

Proof. (i) and (iii) are immediate. For (ii), note that $\text{Val}(\varphi)$ is a concave function since it is the infimum of affine functions. Thus $\text{Val}(\varphi^{-1}) = -\text{Val}(\varphi)$ is both convex and concave and in particular both $\text{Val}(\varphi)$ and $\text{Val}(\varphi^{-1})$ are affine. \square

Example 5.7. The integral affine structure on \mathbb{R}^n can be then be described as the sheaf of functions of the form $\text{Val}(\varphi)$ for $\varphi \in \text{trop}_*(\mathcal{O}_{\mathbb{G}_m^n}^\times)$.

Since $\text{Val}(\varphi)$ is an affine function, we can make the following definition:

Definition 5.8. Given $V \subseteq \mathbb{R}^n$ as above, let $d\text{Val}(\varphi) : V \rightarrow \check{\Lambda}$ be the derivative of $\text{Val}(\varphi)$. This defines a morphism of sheaves $d\text{Val} : \text{trop}_*(\mathcal{O}_{\mathbb{G}_m^n}^\times) \rightarrow \check{\Lambda}$.

More generally, suppose X is an n -dimensional algebraic variety over K and $\rho : X^{\text{an}} \rightarrow B$ an affinoid torus fibration away from a discriminant locus $\Gamma \subset B$ with $B^{\text{sm}} = B \setminus \Gamma$. Let $U \subset B^{\text{sm}}$ be a connected open set which is homeomorphic to an open set of \mathbb{R}^n and suppose the map ρ locally looks like $\text{trop}^{-1}(V) \subset \mathbb{G}_m^n \rightarrow V \subset B^{\text{sm}}$. On V , the induced integral affine structure can then be described as

$$\text{Aff}_{\mathbb{Z}}(V) = \{\text{Val}(\varphi) : \varphi \in \mathcal{O}_{X^{\text{an}}}^\times(\rho^{-1}(V))\}.$$

Let $\mathcal{F}_1 = \ker(\text{Val}) \subset \rho_*(\mathcal{O}_{X^{\text{an}}}^\times)$; note \mathcal{F}_1 is precisely the sheaf of functions which are zero in the induced integral affine structure on B^{sm} and is naturally an \mathcal{O}_K^\times -module. It can be alternatively described as $\mathcal{F}_1 = \{f \in \rho_*(\mathcal{O}_{X^{\text{an}}}^\times) : |f| = 1\}$. **polar coordinates**

We then have the short exact sequence of sheaves on B^{sm}

$$0 \rightarrow K^\times / \mathcal{O}_K^\times \rightarrow \rho_*(\mathcal{O}_{X^{\text{an}}}^\times) / \mathcal{F}_1 \xrightarrow{d\text{Val}} \check{\Lambda} \rightarrow 0,$$

where $K^\times / \mathcal{O}_K^\times$ embeds into $\rho_*(\mathcal{O}_{X^{\text{an}}}^\times) / \mathcal{F}_1$ as constant functions. Fix an analytic volume form on X^{an} . More precisely, let U be an analytic subspace of X^{an} such that $\rho^{-1}(B^{\text{sm}}) \subseteq U$ and $\Omega \in \Gamma(U, \omega_X^{\otimes n})$ a top degree nowhere vanishing analytic volume form on U . The analytic volume form Ω locally is given by the expression

$$(7) \quad \Omega = \varphi(z_1, \dots, z_n) \bigwedge_{i=1}^n \frac{dz_i}{z_i},$$

where (z_1, \dots, z_n) are local torus coordinates on U^{an} and φ is an invertible function on U .

Definition 5.9. We define $\text{Val}(\Omega) := \text{Val}(\varphi)$. By Lemma 2 in [KS06], this is independent of a choice of coordinates.

As in [KS06], we make the following assumption:

Assumption 5.10. $\text{Val}(\Omega)$ is locally constant *i.e.* $d\text{Val}(\Omega) := d\text{Val}(\varphi) = 0$.

Definition 5.11. Let Ω be a nowhere vanishing analytic volume form on X^{an} . Then define the *residue* $\text{Res}(\Omega) \in K$ of Ω to be the constant term of φ defined in (7).

We make the following observation which will be needed in proving that the definition of residue is well defined.

Lemma 5.12. Let $\Omega = \varphi(z_1, \dots, z_n) \bigwedge_{i=1}^n \frac{dz_i}{z_i}$ with $\varphi = \sum_{I \in \mathbb{Z}^n} \varphi_I z^I$. Consider the volume form

$$\Omega' = \Omega - c \bigwedge_{i=1}^n \frac{dz_i}{z_i}$$

with $c \in K$. Then Ω' is exact if and only if $c = \varphi_{(0, \dots, 0)}$.

Lemma 5.13. The residue is independent (up to sign) of the choice of local torus coordinates.

Proof. Let (z_1, \dots, z_n) and (s_1, \dots, s_n) be two systems of local torus coordinates. Then for each i , we can write

$$s_i = c_i z^{\mathbf{I}_i} (1 + \alpha_i),$$

with $c_i \in K^\times$, $\mathbf{I}_i \in \mathbb{Z}^n$ and $\alpha_i = \alpha_i(z_1, \dots, z_n)$ such that:

- (i) the $n \times n$ -matrix $\mathbf{I} = (\mathbf{I}_1, \dots, \mathbf{I}_n) \in \text{GL}_n(\mathbb{Z})$ is unimodular;
- (ii) the function α_i has no degree zero term and $\text{Val}(1 + \alpha_i) = 0$.

It is easy to see that the scalars c_i and the unimodular transformation \mathbf{I} have no effect on the residue. Thus we can assume the coordinate transformation is of the form

$$z_i \mapsto s_i := z_i(1 + \alpha_i).$$

Assume we can write the volume form Ω as

$$\Omega = \varphi(z_1, \dots, z_n) \bigwedge_{i=1}^n \frac{dz_i}{z_i} = \phi(s_1, \dots, s_n) \bigwedge \frac{ds_i}{s_i}$$

with $\varphi = \sum_{I \in \mathbb{Z}^n} \varphi_I z^I$ and $\phi = \sum_{I \in \mathbb{Z}^n} \phi_I s^I$. By Lemma 5.12, the volume form $\Omega - \varphi_{(0, \dots, 0)} \bigwedge_{i=1}^n \frac{dz_i}{z_i}$ is exact. Define

$$u_i(z_1, \dots, z_n) = \frac{s_i}{z_i} = 1 + \alpha_i.$$

Since the dominant monomial is of degree 0, the (Laurent) series $\log(u_i)$ converges and the form $\frac{du_i}{u_i}$ is exact. Let $[n]$ denote the set $\{1, \dots, n\}$ and consider the expression

$$\begin{aligned}
\bigwedge_{i=1}^n \frac{ds_i}{s_i} &= \bigwedge_{i=1}^n \frac{(1 + \alpha_i)dz_i + z_i d\alpha_i}{z_i(1 + \alpha_i)} \\
&= \bigwedge_{i=1}^n \left(\frac{dz_i}{z_i} + \frac{du_j}{u_j} \right) \\
&= \sum_{I \cup J = [n], I \cap J = \emptyset} \bigwedge_{i \in I} \frac{dz_i}{z_i} \wedge \bigwedge_{j \in J} \frac{du_j}{u_j} \\
&= \bigwedge_{i=1}^n \frac{dz_i}{z_i} + \sum_{\substack{I \cup J = [n], I \cap J = \emptyset \\ J \neq \emptyset}} \bigwedge_{i \in I} \frac{dz_i}{z_i} \wedge \bigwedge_{j \in J} \frac{du_j}{u_j} \\
&= \bigwedge_{i=1}^n \frac{dz_i}{z_i} - \sum_{\substack{I \cup J = [n], I \cap J = \emptyset \\ J \neq \emptyset}} d \left(\log u_{j'} \bigwedge_{i \in I} \frac{dz_i}{z_i} \wedge \bigwedge_{j \in J \setminus j'} \frac{du_j}{u_j} \right).
\end{aligned}$$

Thus the form $\bigwedge_{i=1}^n \frac{ds_i}{s_i} - \bigwedge_{i=1}^n \frac{dz_i}{z_i}$ is exact which implies that

$$\Omega - \varphi_{(0, \dots, 0)} \bigwedge_{i=1}^n \frac{ds_i}{s_i}$$

is also exact. By another application of Lemma 5.12 using the coordinate system (s_1, \dots, s_n) , we have $\varphi_{(0, \dots, 0)} = \phi_{(0, \dots, 0)}$. This completes the proof. \square

Lemma 5.14. Let $\Omega = \varphi(z_1, \dots, z_n) \bigwedge_{i=1}^n \frac{dz_i}{z_i}$ be a non-vanishing analytic volume form on X^{an} such that $\text{Val}(\Omega)$ is locally constant. Then $\text{Res}(\Omega) \neq 0$.

Proof. Since Ω is non-vanishing, we have $\exp(-\text{Val}(\Omega)) = \exp(-\text{Val}(\varphi)) = |\varphi| = |\varphi_{(0, \dots, 0)}|$ where in the last equality we use the constant norm assumption $d\text{Val}(\Omega) = 0$. Since Ω is non-vanishing, $\text{Res}(\Omega) \neq 0$. \square

Theorem 5.15. [KS06, Theorem 4] There exists a K -affine structure Aff_K on B^{sm} which is compatible with the integral affine structure $\text{Aff}_{\mathbb{Z}}$ on B^{sm} i.e. there is a diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K^\times & \longrightarrow & \text{Aff}_K & \longrightarrow & \check{\Lambda} \longrightarrow 0 \\
& & \downarrow v_K & & \downarrow & & \downarrow \text{id} \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \text{Aff}_{\mathbb{Z}} & \longrightarrow & \check{\Lambda} \longrightarrow 0
\end{array}$$

where the rows are exact.

Proof. We define the following multiplicative residue map¹

$$\text{Res}_\Omega : \mathcal{F}_1 \rightarrow \mathcal{O}_K^\times$$

¹This map is designed to mimic the notion of residue in complex analysis.

where given an element $f \in \mathcal{F}_1$, after a choice of local torus coordinates, we can write $f = a(1+r)$ with $a \in \mathcal{O}_K^\times$ and $|r| < 1$. Then

$$\text{Res}_\Omega(f) = a \exp\left(\frac{\text{Res}(\Omega \log(1+r))}{\text{Res}(\Omega)}\right).$$

Note Res_Ω remains unchanged if we scale Ω to $c\Omega$ with $c \in K^\times$.

Claim: Res_Ω is well defined *i.e.* is independent of the non-unique factorisation of f .

Proof of Claim: Let (z_1, \dots, z_n) and (s_1, \dots, s_n) be two systems of local torus coordinates and suppose

$$a(1+r(z_1, \dots, z_n)) = b(1+r'(s_1, \dots, s_n)).$$

Since $a, b \in \mathcal{O}_K^\times$, we can write $a = a'(1+\alpha)$ and $b = b'(1+\beta)$ where $a', b' \in \mathbb{C}^\times$ and $1+\alpha, 1+\beta \in 1+t\mathbb{C}[[t]]$. In particular, $\log(a(1+r))$ is well defined and given by $\log(a) + \log(1+\alpha) + \log(1+r)$ and similarly for $\log(b(1+r'))$. Then

$$\text{Res}(\Omega \log(a(1+r))) = \text{Res}(\Omega \log(b(1+r')))$$

since Res is independent of the choice of local torus coordinates by Lemma 5.13. We can write both sides of the equation as

$$\begin{aligned} & \log(a')\text{Res}(\Omega) + \log(1+\alpha)\text{Res}(\Omega) + \text{Res}(\Omega \log(1+r)) \\ &= \log(b')\text{Res}(\Omega) + \log(1+\beta)\text{Res}(\Omega) + \text{Res}(\Omega \log(1+r')). \end{aligned}$$

Dividing by $\text{Res}(\Omega) \neq 0$ and exponentiating proves the claim. \square

We return to the proof of the proposition. We have a short exact sequence

$$(8) \quad 0 \longrightarrow K^\times \longrightarrow \rho_*(\mathcal{O}_{X^{\text{an}}}^\times) / \ker \text{Res}_\Omega \longrightarrow \check{\Lambda} \longrightarrow 0$$

which defines a K -affine structure on B^{sm} compatible with the integral affine structure on B^{sm} . Indeed, by construction, $\ker \text{Res}_\Omega$ is a subsheaf of \mathcal{F}_1 whose elements restrict to 0 on $\text{Aff}_\mathbb{Z}$ and we recover the usual integral affine structure by applying Val to $\rho_*(\mathcal{O}_{X^{\text{an}}}^\times) / \ker \text{Res}_\Omega$. \square

Remark 5.16. In Theorem 5.15, given a non-vanishing analytic volume form, we constructed an explicit K -affine structure. For abelian varieties, the short exact sequence defining the K -affine structure splits and thus there is a unique K -affine structure. For other varieties, the possible K -affine structures are controlled by $\text{Ext}^1(\check{\Lambda}, K^\times)$. The main result of this paper is that the K -affine structure of Theorem 5.15 can be used to compute periods - it is an interesting question to ask if other K -affine structures have geometric meaning.

Example 5.17. As we noted in Example 5.2, the K -affine structure on \mathbb{R}^n is split. The splitting of the K -affine structure in Theorem 5.15 is given by the section $\check{\Lambda} \rightarrow \text{Aff}_K$ which sends a character (where we identify $\check{\Lambda}$ with the character lattice) to the corresponding monomial in $\text{trop}_* \mathcal{O}_{\mathbb{G}_m^{\text{an}}}^\times$.

The following example will be very important when we begin to compute what the K -affine structure Aff_K from Theorem 5.15 looks like locally for log Calabi-Yau surfaces **and more generally for cluster transformations**.

Example 5.18. Let $V \subseteq B^{\text{sm}}$ be a connected open subset such that on V , ρ locally looks like trop . Let $\mu \in \mathcal{O}_K^\times$ and consider the element $\mu + z \in \rho_*(\mathcal{O}_{X^{\text{an}}}^\times)(V)$ for z a local torus coordinate on $\pi^{-1}(V)$ with $|z| < 1$ on V . Then $\text{Val}(\mu + z) = 0$ and writing $\mu + z = \mu \left(1 + \frac{z}{\mu}\right)$ we see that $\text{Res}_\Omega \left(1 + \frac{z}{\mu}\right) = 1$. Thus $\mu + z \equiv \mu$ in $\rho_*(\mathcal{O}_{X^{\text{an}}}^\times) / \ker \text{Res}_\Omega$.

Example 5.19. Any element f which can be written as a product of power series will be an element of $\ker \text{Res}_\Omega$. For example,

$$f = \frac{1}{1-tz} \cdot \frac{1}{1+tz^{-1}} \in \ker \text{Res}_\Omega.$$

Notation 5.20. Throughout this paper, we will denote the explicit K -affine structure constructed in Proposition 5.15 by Aff_K .

5.2. Čech cohomology. A K -affine structure describes an element in $\text{Ext}^1(\check{\Lambda}, K^\times) \cong H^1(B^{\text{sm}}, \Lambda \otimes K^\times)$. We make this isomorphism more explicit now. Dualising and tensoring the short exact sequence defining the K -affine structure by K^\times which is a flat \mathbb{Z} -module, we have the short exact sequence

$$(9) \quad 0 \rightarrow \Lambda \otimes K^\times \rightarrow \mathcal{H}\text{om}(\text{Aff}_K, K^\times) \rightarrow \text{Hom}(K^\times, K^\times) \rightarrow 0.$$

Taking long exact sequences, we have

$$\begin{aligned} 0 \rightarrow H^0(B^{\text{sm}}, \Lambda \otimes K^\times) \rightarrow H^0(B^{\text{sm}}, \mathcal{H}\text{om}(\text{Aff}_K, K^\times)) \rightarrow H^0(B^{\text{sm}}, \text{Hom}(K^\times, K^\times)) \\ \xrightarrow{\delta} H^1(B^{\text{sm}}, \Lambda \otimes K^\times) \rightarrow H^1(B^{\text{sm}}, \mathcal{H}\text{om}(\text{Aff}_K, K^\times)) \rightarrow \dots \end{aligned}$$

The corresponding element in $H^1(B^{\text{sm}}, \Lambda \otimes K^\times)$ is the image, under the connecting homomorphism δ , of the identity element in $\text{Hom}(K^\times, K^\times)$.

Remark 5.21. We can also associate the cohomology class $[\text{Aff}_\mathbb{Z}] \in H^1(B^{\text{sm}}, \check{\Lambda}_\mathbb{R})$ to the integral affine structure on B^{sm} . This class, called the *radiance obstruction class*, was introduced in [GH84]. The radiance obstruction class has been used recently in studying the notion of tropical periods [Yam22], where an interpretation is given in terms of logarithmic Hodge theory.

We will explicitly compute the K -affine structure for the case of log Calabi-Yau surfaces using Čech cohomology in §5.4. By [Sta24, Tag 03F7], given an open cover \mathcal{U} such that the connected components of every finite intersection are contractible, then Čech cohomology agrees with sheaf cohomology.

We now describe how to compute the image of the identity element in $\text{Hom}(K^\times, K^\times)$ via a diagram chase. We assume that the open cover \mathcal{U} is refined enough such that on each open $U \in \mathcal{U}$, the local system $\Lambda|_U$ is constant. The Čech double complex for the open cover \mathcal{U} is as follows.

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_{U \in \mathcal{U}} (U, \Lambda \otimes K^\times) & \longrightarrow & \prod_{U \in \mathcal{U}} (U, \mathcal{H}\text{om}(\text{Aff}_K, K^\times)) & \xrightarrow{\text{rest}} & \prod_{U \in \mathcal{U}} (U, \text{Hom}(K^\times, K^\times)) \longrightarrow 0 \\ & & \downarrow d_0 & & \downarrow d'_0 & & \downarrow d''_0 \\ 0 & \longrightarrow & \prod_{U, U' \in \mathcal{U}} (U \cap U', \Lambda \otimes K^\times) & \longrightarrow & \prod_{U, U' \in \mathcal{U}} \Gamma(U \cap U', \mathcal{H}\text{om}(\text{Aff}_K, K^\times)) & \longrightarrow & \prod_{U, U' \in \mathcal{U}} \Gamma(U \cap U', \text{Hom}(K^\times, K^\times)) \longrightarrow 0 \\ & & \downarrow d_1 & & \downarrow d'_1 & & \downarrow d''_1 \\ & & \dots & & \dots & & \dots \end{array}$$

Let $\alpha = (\text{id}, \cdot, \cdot, \text{id}) \in \prod_U \Gamma(U, \text{Hom}(K^\times, K^\times))$ where $\text{id} : K^\times \rightarrow K^\times$ is the identity element in $\text{Hom}(K^\times, K^\times)$. A lift of the element $\text{id} \in \Gamma(U, \text{Hom}(K^\times, K^\times))$ to $\Gamma(U, \mathcal{H}\text{om}(\text{Aff}_K, K^\times))$ is given

as follows: for each U , we have $H^1(U, \Lambda \otimes K^\times) = 0$ since U is contractible and Λ is a constant sheaf. In particular, the short exact sequence

$$0 \longrightarrow K^\times|_U \longrightarrow \text{Aff}_K|_U \longrightarrow \check{\Lambda}|_U \longrightarrow 0$$

splits and there is a section $s_U : \text{Aff}_K|_U \rightarrow K^\times$. By construction, this is a lift of $\text{id} : K^\times \rightarrow K^\times$ on U .

5.3. Abelian varieties. Our first application of K -affine structures will be to abelian varieties. In particular, we will see that the space of K -affine structures completely classifies uniformisable abelian varieties.

Let us first review the theory of abelian varieties in the sense of Berkovich [Ber90, §6.5]. Let A be an n -dimensional abelian variety over K . Then there exists a canonical subspace $\Delta(A) \subseteq A^{\text{an}}$, called the skeleton, and a continuous retraction $\rho_A : A^{\text{an}} \rightarrow \Delta(A)$. This subspace is constructed from a non-archimedean theory of uniformisation. We will only be interested in the case that A is maximally degenerate *i.e.* the identity component of the special fibre of the Néron model of A is an algebraic torus.

Assume A is maximally degenerate and let $e \in A$ be the identity. The universal pointed covering space of (A^{an}, e) with respect to the Berkovich topology is the analytification of a split n -dimensional torus T with character lattice M and cocharacter lattice $N = M^\vee$. In particular, there is a morphism of Berkovich analytic spaces

$$\pi : T^{\text{an}} \longrightarrow A^{\text{an}}$$

whose kernel $L = \ker \pi$ is a lattice in $T(K)$. The tropicalisation map trop of T is

$$\begin{aligned} \text{trop} : T^{\text{an}} &\longrightarrow N_{\mathbb{R}} \\ |\cdot|_x &\longmapsto (M \rightarrow \mathbb{R}, m \mapsto -\log |m|_x). \end{aligned}$$

Moreover, we have that $\text{trop}(L)$ is an n -dimensional lattice in $N_{\mathbb{R}}$, and by definition $\Delta(A)$ is the image of the skeleton of T under π . There is a Cartesian diagram of topological spaces

$$\begin{array}{ccc} T^{\text{an}} & \xrightarrow{\text{trop}} & N_{\mathbb{R}} \\ \downarrow \pi & & \downarrow \\ A^{\text{an}} & \xrightarrow{\rho_A} & N_{\mathbb{R}}/\text{trop}(L) \end{array}$$

such that ρ_A sends $\Delta(A)$ homeomorphically onto $N_{\mathbb{R}}/\text{trop}(L)$. In particular, $\Delta(A)$ is a real n -dimensional torus, ρ_A is an n -dimensional affinoid torus fibration and the integral affine structure on $\Delta(A)$ coincides with the quotient affine structure on $N_{\mathbb{R}}/\text{trop}(L)$.

We will extend the class of Berkovich analytic spaces we work with in this section to include spaces that do not necessarily arise from the analytification of an algebraic variety. We

Definition 5.22. A K -abeloid variety is a group object in the category of Berkovich K -analytic spaces which is smooth, proper and connected.

Remark 5.23. A K -abeloid variety is the analogue of a complex torus in complex geometry; that is, not every abeloid variety arises as the analytification of an abelian variety as not every complex torus is necessarily algebraic. The conditions for an abeloid variety to come from the analytification of an algebraic variety are discussed in [Lüt16].

Definition 5.24. [Lüt16, page 344] A *formal Néron model* of a K -abeloid variety A is the largest open analytic subgroup \bar{A} which admits a smooth quasi-compact formal model over R .

Just as we restricted to maximally degenerate abelian varieties above, we will only work with a special case of an abeloid variety. Let $T = \mathbb{G}_m^n$ and

$$T_{\circ}^{\text{an}} := \{x \in T^{\text{an}} : |\chi^m|_x = 1 \text{ for all } m \in M\}.$$

Theorem 5.25 ([Lüt16]). Let A be an n -dimensional K -abeloid variety with a formal Néron model of A whose identity component is isomorphic to T_{\circ}^{an} . Let $\Phi : \bar{A} \rightarrow A$ denote the open embedding. Then there exists an analytic group homomorphism $\Psi : T^{\text{an}} \rightarrow A$ such that the restriction to the identity component of \bar{A} is the identity. Then $\ker(\Psi)$ is a discrete subgroup of $T(K)$ and $A \cong T^{\text{an}}/\ker(\Psi)$. ??

Definition 5.26. An n -dimensional K -abeloid variety A with formal Néron model whose identity component is isomorphic to T_{\circ}^{an} shall be referred to as a *toric K -abeloid variety*. In [KS06], a toric abeloid variety is referred to as a Tate torus.

The converse of Theorem ?? More precisely, given an n -dimensional lattice $L \subset T(K)$, one can construct the quotient T^{an}/L where the action of L on T^{an} is induced by the action of $T(K)$ on T^{an} . We check properness below:

Lemma 5.27. Let L be a n -dimensional lattice in $T(K)$ and consider the analytic space $A = T^{\text{an}}/L$. Then A is proper.

Proof. Since L is a full rank lattice in $T(K)$, it is immediate that $\mathbb{R}^n/\text{trop}(L)$ is compact. Furthermore, the induced retraction map $\rho : A \rightarrow \mathbb{R}^n/\text{trop}(L)$ is topologically proper and thus A is compact.

Let $P \subset \mathbb{R}^n$ denote a n -dimensional rational polytope such that P is contained in a fundamental domain of $\mathbb{R}^n/\text{trop}(L)$. Then $U_P := \text{trop}^{-1}(P)$ is an affinoid subdomain (in fact, a Weierstrass domain) of T^{an} .

These subsets form an atlas of charts of A and moreover A can be covered by finitely many of these charts since A is compact. For any affinoid domain of the form above, let P' be a rational polytope with P contained in the interior of P' . Let $K\langle U_P \rangle$ (respectively $K\langle U_{P'} \rangle$) denote the corresponding affinoid algebra of U_P (respectively $U_{P'}$). Then we have the commutative diagram

$$\begin{array}{ccc} K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} & \longrightarrow & K\langle U_P \rangle \\ \uparrow & & \uparrow \\ K\{s_1^{-1}T_1, \dots, s_n^{-1}T_n\} & \longrightarrow & K\langle U_{P'} \rangle \end{array}$$

with $r_i < s_i$ for all i . Let $\varphi : U_P \rightarrow \mathcal{M}(K\{s_1^{-1}T_1, \dots, s_n^{-1}T_n\})$ be the induced map of affinoid domains obtained by taking the spectra of the diagram above. By the commutative diagram, for every $y \in U_P$ we have $|T_i(\varphi(y))| = r_i < s_i$. This proves A has no boundary and is compact; in particular, A is proper. \square

Corollary 5.28. The analytic space $A = T^{\text{an}}/L$ is an abeloid variety.

Let $A = T^{\text{an}}/L$ be a toric abeloid variety over K . We have the following commutative diagram of topological spaces

$$\begin{array}{ccc}
T^{\text{an}} & \xrightarrow{\pi} & A \\
\downarrow \text{trop} & & \downarrow \rho_A \\
\mathbb{R}^n & \xrightarrow{q} & \mathbb{R}^n / \text{trop}(L).
\end{array}$$

The lattice L acts on the K -affine structure $\text{Aff}_K^T := \check{\Lambda} \oplus K^\times$ on \mathbb{R}^n (Example 5.17) via the K^\times summand. Indeed, the lattice L acts on $\text{trop}_* \mathcal{O}_{T^{\text{an}}}^\times$ by scaling the torus coordinates accordingly. The invariant sheaf $(\text{Aff}_K^T)^L$ of this action descends to a sheaf on $\Delta(A) := \mathbb{R}^n / \text{trop}(L)$ and is isomorphic to $\rho_*(\mathcal{O}_A^\times)$. Then $(\text{Aff}_K^T)^L$ defines a K -affine structure on $\Delta(A)$:

$$0 \rightarrow K^\times \rightarrow (\text{Aff}_K^T)^L \rightarrow \check{\Lambda} \rightarrow 0.$$

Thus the lattice L determines a K -affine structure on $\Delta(A)$. In the other direction, a K -affine structure on $\Delta(A)$ is an element of $H^1(\Lambda \otimes K^\times) \cong \text{Hom}(H_1(\Delta(A)), (K^\times)^n)$ and thus determines a lattice embedding $L \rightarrow T(K)$. We then have the following result which is stated in [KS06] without proof.

Theorem 5.29. The groupoid of toric abeloid varieties and the isomorphisms between them is equivalent to the groupoid of K -affine structures on real flat tori.

Proof. It remains to prove the equivalence of the isomorphisms in the groupoids. Indeed, the automorphism group of a toric abeloid K -variety $A = T^{\text{an}}/L$ is given by the automorphism group of the lattice L .

The automorphisms of a K -affine structure $(\text{Aff}_K^T)^L$ pull back to automorphisms of a K -affine structure on \mathbb{R}^n . The automorphisms of the K -affine structure on \mathbb{R}^n are given by $\text{GL}_n(\mathbb{Z}) \ltimes (K^\times)^n$ since it is a $\text{GL}_n(\mathbb{Z}) \ltimes (K^\times)^n$ -torsor. In particular, an automorphism of $(\text{Aff}_K^T)^L$ must pull back to an automorphism of the lattice L via q . This completes the proof. \square

Remark 5.30. We note that the K -affine structure constructed in Theorem 5.15 coincides with the $(\check{\Lambda} \oplus K^\times)^L$ above. To see this, let Ω be the T -invariant analytic volume form, which in torus coordinates for T is given by $\bigwedge_{i=1}^n \frac{dz_i}{z_i}$. This descends to a nowhere-vanishing analytic volume form on A . Note $\text{Val}(\Omega)$ satisfies the constant norm assumption (5.10). There is a natural surjection

$$\Phi : (\text{Aff}_K^T)^L \rightarrow \text{Aff}_K = \rho_*(\mathcal{O}_A^\times) / \ker(\text{Res}_\Omega)$$

given by the quotient map. By construction, $\ker(\text{Res}_\Omega) \subset \mathcal{F}_1$ on $\Delta(A)$ consists of elements f which are zero in $\text{Aff}_{\mathbb{Z}}$ on $\Delta(A)$. In particular, they pull back to functions which are zero in $\text{Aff}_{\mathbb{Z}} = \check{\Lambda} \oplus \mathbb{R}$ on \mathbb{R}^n and thus are constant functions $f = c \in \mathcal{O}_K^\times$ in $(\check{\Lambda} \oplus K^\times)^L$. Since $f \in \ker(\text{Res}_\Omega)$, we must have $c = 1$. Thus we have an equivalence of short exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & K^\times & \longrightarrow & (\check{\Lambda} \oplus K^\times)^L & \longrightarrow & \check{\Lambda} \longrightarrow 0 \\
& & \downarrow \text{id} & & \downarrow \Phi & & \downarrow \text{id} \\
0 & \longrightarrow & K^\times & \longrightarrow & \text{Aff}_K & \longrightarrow & \check{\Lambda} \longrightarrow 0.
\end{array}$$

5.4. Log Calabi-Yau surfaces. In this section, we describe Aff_K associated to the non-archimedean SYZ fibration $\rho = \rho_{(\mathcal{Y}, \mathcal{D})}$ with $(\mathcal{Y}, \mathcal{D})$ the snc log model for $U = Y \setminus D$ from Proposition 3.34 and compute the monodromy representation (5.4) of Aff_K ; as a corollary, we will recover the monodromy of the integral affine structure (3.45). Recall the construction of $\text{Aff}_K = \frac{\rho_*(\mathcal{O}_{U^{\text{an}}}^\times)}{\ker \text{Res}_\Omega}$ depends on a volume form Ω . In the case of log Calabi-Yau surfaces, there is a distinguished volume form

given by the pullback of the standard torus invariant volume form on \bar{Y} . This volume form has simple poles along D and is unique up to scaling. It is immediate that Ω satisfies the constant norm assumption (5.10).

To compute the monodromy, we will utilise the proof of Proposition 3.37 to construct a basis of affinoid torus coordinates for each local toric chart and compute the composition of the transition functions for Aff_K .

We first introduce the setup and notation needed to carry out the computation. We fix an orientation on D where the node $D_{i-1} \cap D_i$ is labelled ∞_{i-1} on D_{i-1} and 0_i on D_i . This naturally induces an orientation on the skeleton $\text{Sk}(U)$ and in particular on the boundary of each irreducible component in the special fibre as illustrated in Figure 14. Let τ_C be a 1-dimensional stratum in $\text{Sk}(U)$ corresponding to C in $\mathcal{Y}_k + \mathcal{D}$.

Convention 5.31. The anticlockwise direction around a vertex corresponds to the positive orientation.

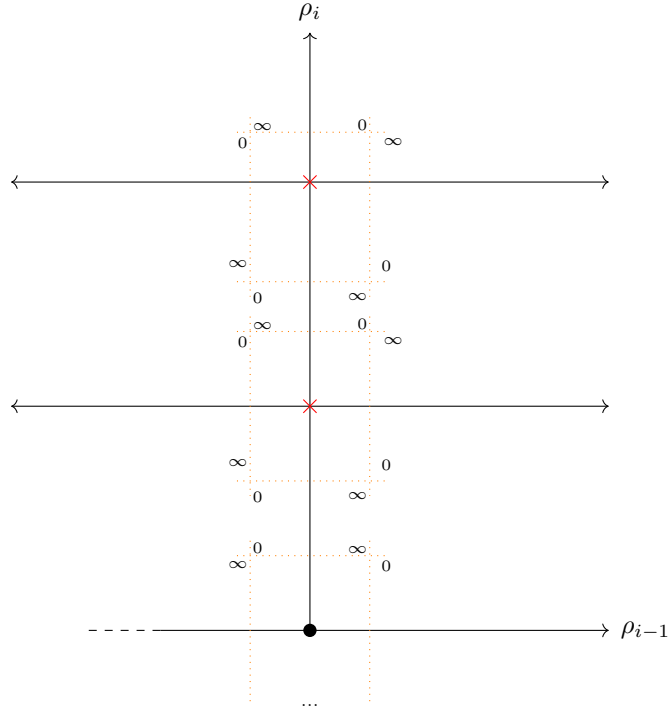


FIGURE 14. An orientation of the boundary D induces an orientation of the boundary Δ_X for every irreducible component X of \mathcal{Y}_k .

Let $X = Y_j^i$ be a non-toric irreducible component of \mathcal{Y}_k with boundary $\Delta_X = C_1 + C_2 + C_3 + C_4$ and denote the corresponding vertex $v_X = v_j^i$ in $\text{Sk}(U)$. Suppose that the non-toric exceptional curve E_X supported on X corresponds to a non-toric exceptional curve E_j^i on Y whose centre is on \bar{D}_i . Without loss of generality, let C_1 denote the boundary divisor containing the centre of the non-toric blowup on X . Let $H^{(m)}$ denote the prime component of the divisor $\mathcal{Y}_k + \mathcal{D}$ containing

C_m which is not equal to X . The induced orientation on X determines the divisors X_0 and X_∞ for each C_m as defined in §3.7; we do not write the superscripts for X_0 and X_∞ as it will be clear from context which C_m we are working with.

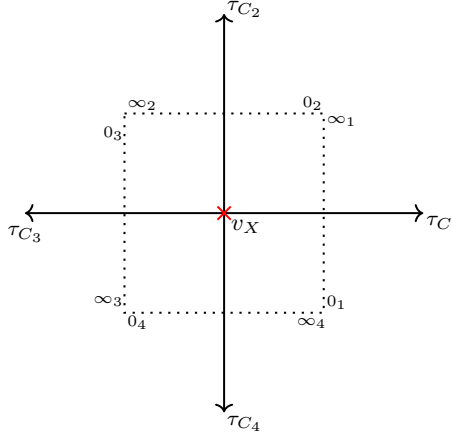


FIGURE 15. Orientation on Δ_X

For each C_m , we will give expressions for generators of

$$\begin{aligned} \mathcal{L}_0^{(m)} &:= \mathcal{O}_{\widehat{\mathcal{Y}}/C_m}(X_\infty - X_0) \\ \mathcal{L}_h^{(m)} &:= \mathcal{O}_{\widehat{\mathcal{Y}}/C_m}(-H^{(m)} - b_h X_\infty) \end{aligned}$$

where $b_h = -(C_m \cdot H^{(m)})$ in terms of the pullback of torus coordinates on $\mathbb{G}_{m,K}^2 \subset \overline{Y}$.

Notation 5.32. Let $s_0^{(m)}, s_h^{(m)}$ be generators of $\mathcal{L}_0^{(m)}$ and $\mathcal{L}_h^{(m)}$ respectively.

Let $(\mathcal{Y}^{(1)}, \mathcal{D}^{(1)})$ denote the Looijenga pair over R with good reduction from Assumption 3.22 such that $(\mathcal{Y}^{(1)}, \mathcal{D}^{(1)}) \times_R K = (Y, D)$. Consider the fan $\overline{\Sigma}$ of the toric model $(\overline{Y}, \overline{D})$ of (Y, D) and suppose the divisor \overline{D}_i corresponds to the ray $\overline{\rho}_i$ of $\overline{\Sigma}$; let \mathbf{v}_i denote the primitive integral generator of $\overline{\rho}_i$.

Consider the pair of torus coordinates $(x, z) := (\chi^{\mathbf{v}_i^\vee}, \chi^{\mathbf{v}_i^\vee})$ on Y and suppose the zero locus of $x + \mu_j^i$ intersected with \overline{D}_i is the centre of the non-toric blowup with exceptional curve E_j^i . We note that the zero locus of z in Y contains all the non-toric exceptional curves arising from non-toric blowups with centre on \overline{D}_i .

5.5. Two technical lemmas. We need the following two technical lemmas to justify the expressions of $s_h^{(m)}$ for $m = 1, 3$ as these are the strata containing the centre of a non-toric blowup. In particular, we need to keep track of how the birational modifications of §3.3 change the affinoid torus coordinates of the induced affinoid torus fibration. The first lemma deals with the case when we move all the non-toric exceptional curves with centre on \overline{D}_i to the new irreducible component given by blowing up $\mathcal{D}_i^{(1)} \times_R \mathbb{C}$. **prove these lemmas using toric geometry**

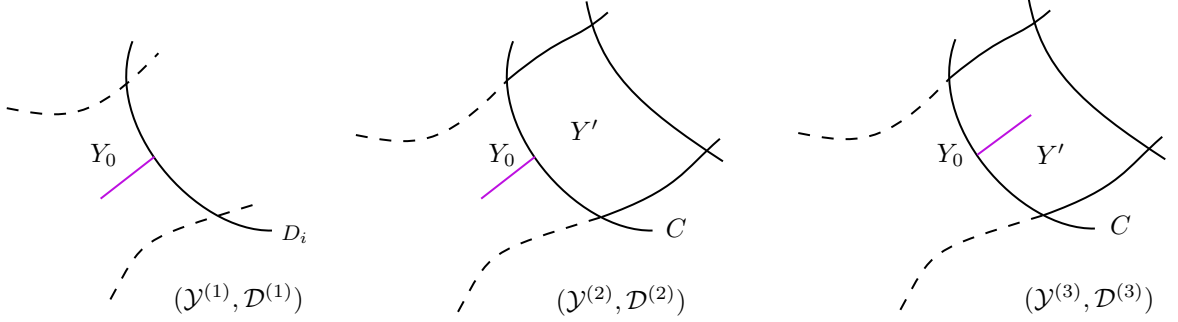


FIGURE 16. Setup for Lemma 5.33

Lemma 5.33. Let (Y, D) be a generic Looijenga pair satisfying Assumption 3.22 and $(\mathcal{Y}^{(1)}, \mathcal{D}^{(1)})$ as above. Let $(\mathcal{Y}^{(2)}, \mathcal{D}^{(2)})$ denote the pair over R given by blowing up the subvariety $\mathcal{D}_i^{(1)} \times_R \mathbb{C}$. Denote the new irreducible component by Y' . Let $(\mathcal{Y}^{(3)}, \mathcal{D}^{(3)})$ denote the pair over R obtained by applying Construction 3.27 (i) to all the exceptional curves with centre on D_i i.e. flop all the non-toric exceptional curves with centre on $D_i \times_R k$ to the newly produced irreducible component Y' . The divisor X_∞ is given by the horizontal divisor corresponding to D_{i+1} . Then away from X_∞ , we have

$$s^{(3)} = s^{(2)} \prod_j (x + \mu_j^i)$$

where $s^{(2)}$ is a generator of the line bundle $\mathcal{O}_{\widehat{\mathcal{Y}^{(2)}/C}}(-Y' - b_{Y'}^{(2)} X_\infty)$, $s^{(3)}$ is a generator of the line bundle $\mathcal{O}_{\widehat{\mathcal{Y}^{(3)}/C}}(-Y' - b_{Y'}^{(3)} X_\infty)$, $b_{Y'}^{(n)} = -(C \cdot Y')_{\mathcal{Y}^{(n)}}$ and C is the double curve $Y \cap Y'$ depicted in Figure 16.

Proof. We can check the equality assuming there is a single non-toric blowup on \overline{D}_i given by the equation $x + \mu$. Then locally, the Looijenga pair $(\mathcal{Y}^{(1)}, \mathcal{D}^{(1)})$ is given by the equation

$$Z((x + \mu)Y = zX) \subseteq \mathbb{A}_{x,z}^2 \times \mathbb{P}^1_{(X:Y)}$$

where $\mathbb{A}^2 \times \mathbb{P}^1$ is defined over $R = \mathcal{O}_K$ and $\mu \in \mathcal{O}_K^\times$. Blowing up the ideal (Y, t) corresponds to blowing up the subvariety $\mathcal{D}_i^{(1)} \times_R \mathbb{C}$ and is given by the equations

$$Z((x + \mu)Y = zX, YX' = tY') \subseteq \mathbb{A}_{x,z}^2 \times \mathbb{P}^1_{(X:Y)} \times \mathbb{P}^1_{(X':Y')}.$$

Away from X_∞ , $s^{(2)}$ is given by the equation $Y' = 0$ and thus can be written as $\frac{z}{x+\mu}$ in torus coordinates. The ideal $(x + \mu, X')$ defines the non-toric exceptional curve $F_\mu \subset Y$; blowing up at this ideal gives

$$\begin{aligned} Z((x + \mu)Y = zX, YX' = tY', (x + \mu)Y'' = X'X'') \\ \subseteq \mathbb{A}_{x,z}^2 \times \mathbb{P}^1_{(X:Y)} \times \mathbb{P}^1_{(X':Y')} \times \mathbb{P}^1_{(X'':Y'')}. \end{aligned}$$

The contraction of the ruling is then given by

$$Z((x + \mu)Y'' = X'X'', zX' = tY') \subseteq \mathbb{A}_{x,z}^2 \times \mathbb{P}^1_{(X':Y')} \times \mathbb{P}^1_{(X'':Y'')}.$$

Away from X_∞ , $s^{(3)}$ is still given by $Y' = 0$ and in torus coordinates is cut out by z . This completes the proof.

□

The next lemma deals with the case when we want to move all but one non-toric exceptional curve off an irreducible component as depicted in Figure 5.5.

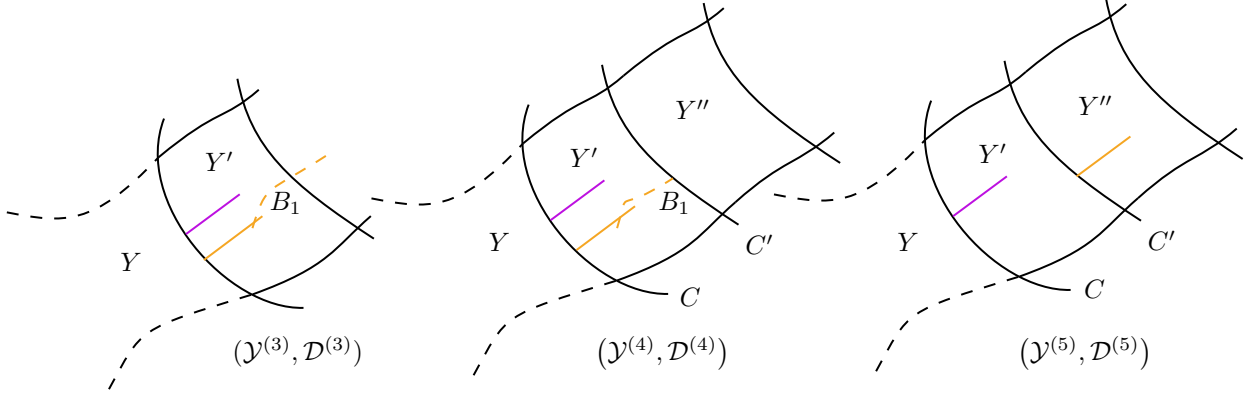


FIGURE 17. Setup for Lemma 5.34

Lemma 5.34. We reuse the notation from Lemma 5.33. Let $B_j \subset Y'$ denote the strict transform of the fibre of Y' which passes through μ_j^i . Let $(\mathcal{Y}^{(4)}, \mathcal{D}^{(4)})$ denote the pair produced by blowing up $\mathcal{D}_i^{(3)} \times_R \mathbb{C}$. Let $(\mathcal{Y}^{(5)}, \mathcal{D}^{(5)})$ denote the pair over R produced by running Construction 3.27 (ii) to all but one B_j i.e. flop all but one B_j , say B_1 , to the new irreducible component Y'' . Then away from X_∞ , we have

$$s^{(4)} \prod_{j \neq 1} (x + \mu_j^i) = t s^{(5)}$$

where $s^{(4)}$ is a generator of the line bundle $\mathcal{O}_{\widehat{\mathcal{Y}^{(4)}/C'}}(-Y'' - b_{Y''}^{(4)} X_\infty)$, $s^{(5)}$ is a generator of the line bundle $\mathcal{O}_{\widehat{\mathcal{Y}^{(5)}/C'}}(-Y'' - b_{Y''}^{(5)} X_\infty)$, $b_{Y''}^{(n)} = -(C' \cdot Y'')_{\mathcal{Y}^{(n)}}$, $C' = Y' \cap Y''$ and

$$r^{(5)} \prod_{j \neq 1} (x + \mu_j^i) = r^{(4)}$$

where $r^{(4)}$ is a generator of the line bundle $\mathcal{O}_{\widehat{\mathcal{Y}^{(4)}/C}}(-Y' - b_{Y'}^{(4)} X_\infty)$, $r^{(5)}$ is a generator of the line bundle $\mathcal{O}_{\widehat{\mathcal{Y}^{(5)}/C}}(-Y' - b_{Y'}^{(5)} X_\infty)$, $b_{Y'}^{(n)} = -(C \cdot Y')_{\mathcal{Y}^{(n)}}$, $C = Y \cap Y'$.

Proof. As in the proof of Lemma 5.33, we can reduce to the case of a single curve B_1 . In this case we run through Construction 3.27 and flop B_1 to the new irreducible component Y'' . Locally, the pair $(\mathcal{Y}^{(3)}, \mathcal{D}^{(3)})$ is given by the set of equations

$$Z((x + \mu)Y'' = X'X'', zX' = tY') \subseteq \mathbb{A}_{x,z}^2 \times \mathbb{P}_{(X':Y')}^1 \times \mathbb{P}_{(X'':Y'')}^1.$$

The stratum $\mathcal{D}_i^{(3)} \times_R \mathbb{C}$ is given by the ideal (Y', t) . Blowing up this ideal gives

$$V((x + \mu)Y'' = X'X'', zX' = tY', Y'V = tW) \subseteq \mathbb{A}_{x,z}^2 \times \mathbb{P}_{(X':Y')}^1 \times \mathbb{P}_{(X'':Y'')}^1 \times \mathbb{P}_{(V:W)}^1.$$

Away from X_∞ , the section $r^{(4)}$ is given by $X' = 0$ which can be written as $\frac{t(x+\mu)}{z}$ and $s^{(4)}$ is given by $W = 0$ which can also be written as $\frac{z}{t}$ in torus coordinates.

The curve B_1 is given by the ideal (X'', V) ; we blow up this ideal which gives

$$\begin{aligned} V((x + \mu)Y'' = X'X'', zX' = tY', Y'V = tW, X''V' = VW') \\ \subseteq \mathbb{A}_{x,z}^2 \times \mathbb{P}_{(X':Y')}^1 \times \mathbb{P}_{(X'':Y'')}^1 \times \mathbb{P}_{(V:W)}^1 \times \mathbb{P}_{(V':W')}^1. \end{aligned}$$

We contract the ruling to produce

$$V((x + \mu)V' = W'W, zX' = tY', Y'V = tW) \subseteq \mathbb{A}_{x,z}^2 \times \mathbb{P}_{(X'':Y'')}^1 \times \mathbb{P}_{(V:W)}^1 \times \mathbb{P}_{(V':W')}^1.$$

Away from X_∞ , $r^{(5)}$ is given by $X' = 0$ and is thus cut out by $\frac{t}{z}$; similarly, $s^{(5)}$ is given by $W = 0$ and is thus cut out by $\frac{z(x+\mu)}{t^2}$. This completes the proof. \square

5.6. Constructing the affinoid torus coordinates. In the notation of Lemmas 5.33 and 5.34, we have that $r^{(4)} = s^{(3)}$ since the construction of the component Y'' still gives an isomorphism $\mathcal{Y}^{(3)}/C \cong \mathcal{Y}^{(4)}/C$. Combining both lemmas, we have the equalities

$$r^{(5)} \prod_{j \neq 1} (x + \mu_j^i) = r^{(4)} = s^{(3)} = s^{(2)} \prod_{j=1}^{k_i} (x + \mu_j^i)$$

which simplifies to

$$(10) \quad r^{(5)} = s^{(2)}(x + \mu_1^i).$$

The section $s^{(2)}$ is a local equation for Y' and thus $s^{(2)} = \frac{z}{(x+\mu_1^i)\dots(x+\mu_{k_i}^i)}$. We will now work with the following pair of coordinates:

$$(x, y) := \left(x, \frac{z}{\prod_{j=1}^{k_i} (x + \mu_j^i)} \right).$$

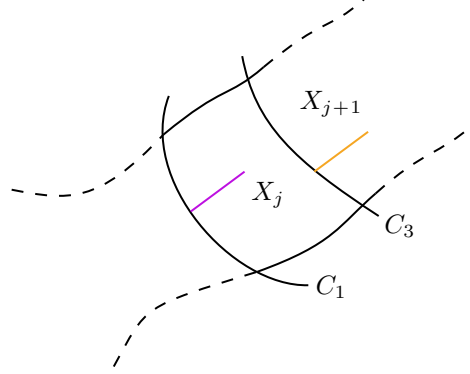
Note that there are now no non-toric exceptional curves in the zero locus of y . With these coordinates, we then have $s^{(2)} = y$ and $s^{(4)} = \frac{y}{t}$.

Recall the vertex $v_j^i \in \rho_i$ is at a distance $j \in \mathbb{N}$ from O (the toric vertex). This means that there are $j - 1$ vertices on ρ_i between O and v_j^i . We also have that the irreducible component $X_j = Y_j^i$ corresponds to the non-toric exceptional curve E_j^i ; this non-toric blowup is given by the equation $x + \mu_j^i$. The irreducible component X_j supports the exceptional divisor corresponding to the blowup of μ_j^i .

We can repeat the argument above to construct the sections for the other irreducible components inductively. Indeed, we have that away from X_∞ ,

$$s_{X_j} = \frac{y}{t^j}(x + \mu_j^i)$$

where $s_{X_j} \in \mathcal{O}_{\widehat{Y}/C_1}(-X_j - b_{X_j}X_\infty)$ is a generator with C_1 depicted below and $b_{X_j} = -(C_1 \cdot X_j)$.



We can put all of this together and now describe the sections for each 1-dimensional stratum emanating from the singular vertices of $\text{Sk}(U)$. We fix the vertex $v_{X_j} \in \rho_i$. Recall $s_0^{(m)}, s_h^{(m)}$ are generators of $\mathcal{L}_0^{(m)}$ and $\mathcal{L}_h^{(m)}$ respectively and let $s_v^{(m)}$ be a generator of the line bundle $\mathcal{O}_{\widehat{\mathcal{Y}/C_m}}(-X_j - b_{X_j} X_\infty)$; note we can choose $s_v^{(m)} = s_{X_j}$ for $m = 1$ and for $m = 1, 3$ we have the relation $s_v^{(m)} s_h^{(m)} = t$, since X_j and $H^{(m)}$ are both vertical divisors and X_0 and X_∞ are horizontal divisors. Similarly, for $m = 2, 4$, we have the relation $s_0^{(m)} s_v^{(m)} = t$. The orientation on X_j is give in Figure 15. Recall that $\mathcal{Y}_0 = \mathcal{Y} \setminus \{c_\infty\}$ where $\mathcal{Y} = \widehat{\mathcal{Y}}/C_m$. Then we have:

(i) along the formal completion of C_1 :

$$(a_1, b_1) := (s_0^{(1)}, s_h^{(1)}|_{\mathcal{Y}_0}) = \left(x, \frac{t^{j+1}}{y(x + \mu_j^i)} \right);$$

(ii) along the formal completion of C_2 :

$$(a_2, b_2) := (s_0^{(2)}, s_h^{(2)}|_{\mathcal{Y}_0}) = \left(\frac{t^{j+1}}{y}, \frac{1}{x} \right);$$

(iii) along the formal completion of C_3 :

$$(a_3, b_3) := (s_0^{(3)}, s_h^{(3)}|_{\mathcal{Y}_0}) = \left(\frac{1}{x}, \frac{y(1 + \frac{\mu_j^i}{x})}{t^{j+1}} \right),$$

where the orientation on C_3 in X_j as in Figure 14 is different from the orientation on C_1 and thus gives rise to the slightly different expression for $s_h^{(3)}$;

(iv) along the formal completion of C_4 :

$$(a_4, b_4) := (s_0^{(4)}, s_h^{(4)}|_{\mathcal{Y}_0}) = \left(\frac{y}{t^{j+1}}, x \right).$$

5.7. Monodromy of K -affine structure. With the expressions above, we can now compute the monodromy of Aff_K for the log Calabi-Yau surface $U = Y \setminus D$.

Proposition 5.35. The monodromy of the K -affine structure around v_{X_j} with respect to the affinoid torus coordinates $(a_2, b_2) = (s_0^{(2)}, s_h^{(2)})$ is

$$\begin{aligned} T_{\text{Aff}_K} : \text{Aff}_K(\text{Star}(\tau_{C_2})) &\longrightarrow \text{Aff}_K(\text{Star}(\tau_{C_2})) \\ a_2 &\longmapsto a_2/\mu_j^i \\ b_2 &\longmapsto a_2 b_2. \end{aligned}$$

Proof. Let γ be a loop around v_{X_j} connecting $v_{C_1}, v_{C_2}, v_{C_3}, v_{C_4}$ and v_{C_1} in that order. As in the integral affine structure computation, we take our charts to be given by $S_m := \text{Star}(\tau_{C_m})$.

The transition map $\varphi_{m,m+1}$ for $\rho_{(\mathcal{Y}, \mathcal{D})}^{-1}(S_m \cap S_{m+1})$ is

$$\begin{array}{ccc} \rho_{(\mathcal{Y}, \mathcal{D})}^{-1}(S_m) & & \rho_{(\mathcal{Y}, \mathcal{D})}^{-1}(S_{m+1}) \\ \cup & & \cup \\ \rho_{(\mathcal{Y}, \mathcal{D})}^{-1}(S_m \cap S_{m+1}) & \xrightarrow{\varphi_{i,i+1}} & \rho_{(\mathcal{Y}, \mathcal{D})}^{-1}(S_m \cap S_{m+1}) \\ x_m & \longmapsto & (x_m)^{p_m} y_m^{-1} \\ y_m & \longmapsto & x_m \end{array}$$

where x_m is a torus coordinate corresponding to $\chi^{v_{C_m}}$ and y_m a torus coordinate corresponding to $\chi^{v_{C_{m+1}}}$ with $p_{m+1} = -(C_{m+1} \cdot H^{(m+1)})$. By the geometry of the model $(\mathcal{Y}, \mathcal{D})$, we know $p_2 = p_4 = 0$ and $p_1 + p_3 = -1$.

The volume form $\Omega = \frac{dx \wedge dz}{xz}$ when restricted to the toric charts S_m is equal to

$$\Omega = -\frac{da_i \wedge db_i}{a_i b_i}.$$

We can then simplify the expressions:

$$b_1 \equiv \frac{t^{j+1}}{\mu_j^i y} \text{ and } b_3 \equiv \frac{y}{t^{j+1}} \text{ in } \text{Aff}_K,$$

since

$$\begin{aligned} \text{Res}_\Omega \left(1 + \frac{a_1}{\mu_j} \right) &= \exp \left(\frac{\text{Res} \left(\Omega \log \left(1 + \frac{a_1}{\mu_j} \right) \right)}{\text{Res}(\Omega)} \right) \\ &= \exp \left(\text{Res} \left(-\frac{a_1 \wedge db_1}{a_1 b_1} \sum_{m=1}^{\infty} \left(-\frac{a_1}{\mu_j} \right)^{m+1} \right) \right) \\ &= 1 \end{aligned}$$

and a similar argument follows for b_3 .

Starting in the chart S_2 and following γ , we track each coordinate change below:

$$\begin{aligned}
(a_2, b_2) &\xrightarrow{\varphi_{2,3}} \left(\frac{1}{b_2}, \frac{1}{a_3} \right) = \left(\frac{1}{a_3}, b_3 \right) \\
&\xrightarrow{\varphi_{3,4}} (b_3, a_3 b_3^{p_3}) = \left(a_4, \frac{a_4^{p_3}}{b_4} \right) \\
&\xrightarrow{\varphi_{4,1}} \left(\frac{1}{b_4}, \frac{1}{a_4 b_4^{p_3}} \right) = \left(\frac{1}{a_1}, \frac{\mu_j^i b_1}{a_1^{p_3}} \right) \\
&\xrightarrow{\varphi_{1,2}} (b_1, \mu^{p_1+p_3} a_1 b_1^{p_1+p_3}) = \left(b_1, \frac{a_1}{\mu_j^i b_1} \right) = \left(\frac{a_2}{\mu_j^i}, a_2 b_2 \right).
\end{aligned}$$

Thus the monodromy of the K -affine structure is given by $a_2 \mapsto a_2/\mu_j^i$ and $b_2 \mapsto a_2 b_2$ as claimed. \square

By construction, the coordinates (a_2, b_2) correspond to $(\chi^{v_{c_1}}, \chi^{v_{c_2}})$ and thus we recover the integral affine structure computation of §3.8 by taking exponents.

6. NON-ARCHIMEDEAN PERIODS

In this section, we will define the non-archimedean period map and prove the main theorem (1.1). In §4, we introduced the domain of the non-archimedean period map as the set of tropical cycles $H_1(\mathrm{Sk}(U), \iota_* \check{\Lambda})$. In §5, we introduced K -affine structures and constructed an explicit one, which we denoted Aff_K . We will now see that this explicit choice in fact computes periods of abelian varieties and log Calabi-Yau surfaces. In both cases, we then apply a Torelli theorem to conclude that the K -affine structure determines the isomorphism type of the underlying variety.

Definition 6.1. [KS06, §7.4.2] Let $\pi : X \rightarrow B$ be a non-archimedean SYZ fibration. Then the smooth locus B^{sm} has the K -affine structure $[\mathrm{Aff}_K]$ constructed in Proposition 5.15. The *non-archimedean period map* is

$$\mathcal{P}^{\mathrm{an}} : H_1(B^{\mathrm{sm}}, \check{\Lambda}) \rightarrow H_0(B^{\mathrm{sm}}, K^\times) = K^\times$$

given by the pairing with $[\mathrm{Aff}_K] \in H^1(B^{\mathrm{sm}}, \Lambda \otimes K^\times)$.

6.1. Periods of abelian varieties. We have already seen that for toric abeloid varieties, the K -affine structure determines the periods of A . In this case, the computation is fairly straightforward as there are no singularities on the base $\Delta(A)$ of the fibration. In this section, we simply verify that the non-archimedean period map still computes the periods.

Let $\{\check{\mathbf{e}}_1, \dots, \check{\mathbf{e}}_n\}$ be a basis of $\check{\Lambda}_{\mathbb{R}^n}$ where $\mathrm{dVal}(z_i) = \check{\mathbf{e}}_i \in \check{\Lambda}$ and let $\mathbf{e}_i \in \Lambda$ denote the dual vector to $\check{\mathbf{e}}_i$ i.e. $\langle \mathbf{e}_i, \check{\mathbf{e}}_j \rangle = \delta_{ij}$. Let $\gamma_{\mathbf{e}_i, \check{\mathbf{e}}_j}$ denote the cycle in $H_1(\Delta(A), \check{\Lambda})$ corresponding to the loop generated by \mathbf{e}_i decorated with the cotangent vector $\check{\mathbf{e}}_j$. Then the pairing of $H_1(\Delta(A), \check{\Lambda})$ with the K -affine structure is given by $\gamma_{\mathbf{e}_i, \check{\mathbf{e}}_j} \mapsto \langle \mathbf{e}_i, L \cdot \check{\mathbf{e}}_j \rangle$. In particular, the non-archimedean period map recovers the lattice L and computes the periods of A . This proves Conjecture ?? for maximally degenerate abelian varieties:

Theorem 6.2. The analytic period map for maximally degenerate abelian varieties agrees with the non-archimedean period map.

6.2. Periods of log Calabi-Yau surfaces.

6.3. K -affine structure. Let (Y, D) be a generic Looijenga pair over K satisfying Assumption 3.22 and write $U = Y \setminus D$. Let $\text{Sk}(U)$ be the skeleton of the model $(\mathcal{Y}, \mathcal{D})$ constructed in Proposition 3.34. We completely described the singular integral affine structure in §3.8 and the monodromy of the K -affine structure $\text{Aff}_K = \frac{\rho_*(\mathcal{O}_{U^{\text{an}}})}{\ker \text{Res}_\Omega}$ in Proposition 5.35.

We begin by computing the Čech cocycle corresponding to the K -affine structure around each vertex corresponding to a non-toric irreducible component in the special fibre. We follow the notation described in §3.8 and §5.4.

In §5.4, we described the transition functions for $\rho_{(\mathcal{Y}, \mathcal{D})}(\mathcal{O}_{U^{\text{an}}}^\times)$ when moving between the charts given by the stars of 1-dimensional strata. We write the transition functions for Aff_K below for moving around a singularity $v_X = v_j^i$ in $\text{Sk}(U)$ with $\mu_X = \mu_j^i$:

$S_1 \cap S_2$	$(s_0^{(2)}, s_h^{(2)}) = \left(\mu_X s_h^{(1)}, \frac{1}{s_0^{(1)}} \right)$
$S_2 \cap S_3$	$(s_0^{(3)}, s_h^{(3)}) = \left(s_h^{(2)}, \frac{1}{s_0^{(2)}} \right)$
$S_3 \cap S_4$	$(s_0^{(4)}, s_v^{(4)}) = \left(s_h^{(3)}, \frac{1}{s_0^{(3)}} \right)$
$S_1 \cap S_4$	$(s_0^{(1)}, s_h^{(1)}) = \left(s_h^{(4)}, \frac{1}{s_0^{(4)} \mu_X} \right)$

TABLE 1. Coordinate transforms for moving around v_X

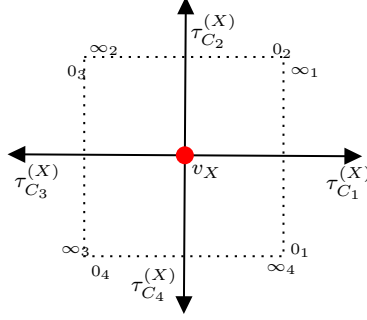
Let $\text{Sk}(U)^{[1]}$ denote the set of 1-dimensional strata of $\text{Sk}(U)$ and $S_\tau = \text{Star}(\tau)$ for $\tau \in \text{Sk}(U)^{[1]}$. We define $S_{\tau, \tau'} = S_\tau \cap S_{\tau'}$. Then we have an open cover \mathcal{U} of $\text{Sk}(U)^{\text{sm}}$ given by $\mathcal{U} = \{S_\tau : \tau \in \text{Sk}(U)^{[1]}\}$. The Čech double complex is

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_\tau \Gamma(S_\tau, \Lambda \otimes K^\times) & \longrightarrow & \prod_\tau \Gamma(S_\tau, \mathcal{H}om(\text{Aff}_K, K^\times)) & \xrightarrow{\text{rest}} & \prod_\tau \Gamma(S_\tau, \text{Hom}(K^\times, K^\times)) \longrightarrow 0 \\
& & \downarrow d_0 & & \downarrow d'_0 & & \downarrow d''_0 \\
0 & \longrightarrow & \prod_{\tau, \tau'} \Gamma(S_{\tau, \tau'}, \Lambda \otimes K^\times) & \longrightarrow & \prod_{\tau, \tau'} \Gamma(S_{\tau, \tau'}, \mathcal{H}om(\text{Aff}_K, K^\times)) & \longrightarrow & \prod_{\tau, \tau'} \Gamma(S_{\tau, \tau'}, \text{Hom}(K^\times, K^\times)) \longrightarrow 0 \\
& & \downarrow d_1 & & \downarrow d'_1 & & \downarrow d''_1 \\
& & \dots & & \dots & & \dots
\end{array}$$

and we consider the lift $\beta := (\sigma_\tau)_\tau \in \prod_\tau \Gamma(S_\tau, \mathcal{H}om(\text{Aff}_K, K^\times))$ of $\alpha := (\text{id})_\tau \in \prod_\tau \Gamma(S_\tau, \text{Hom}(K^\times, K^\times))$ as described in §5.2.

Notation 6.3. For each 1-dimensional stratum τ of $\text{Sk}(U)$ and corresponding sections s_0, s_v , let $\check{\mathbf{e}}_0 := \text{dVal}(s_0) \in \check{\Lambda}|_{S_\tau}$ and $\check{\mathbf{e}}_v = \text{dVal}(s_v) \in \check{\Lambda}|_{S_\tau}$ and $\mathbf{e}_0, \mathbf{e}_v \in \Lambda|_{S_\tau}$ denote the dual basis to $\check{\mathbf{e}}_0, \check{\mathbf{e}}_v$ respectively. The local toric chart S_τ is implicit in the notation; we will make it clear which chart we will be considering these sections in.

Notation 6.4. Given a vertex $v_X = v_j^i$ corresponding to a non-toric irreducible component X , let C_m denote the irreducible components of the boundary of $X = Y_j^i$ and $\tau_{C_m}^{(X)}$ the corresponding



edge/ray as depicted in the figure above. Continuing with our previous convention, we will assume that C_1 is the boundary divisor containing the centre of the non-toric blowup on X . When it is clear, we will write $\tau_{C_m} = \tau_{C_m}^{(X)}$.

To completely describe the differential, there are two cases to consider:

- (i) if $\tau = \tau_{C_m}^{(X)}$ and $\tau' = \tau_{C_{m'}}^{(X)}$ intersect at a common vertex v_X . In this case, Table 1 describes the transition functions for the intersection $S_{\tau, \tau'}$;
- (ii) if $\tau = \tau_{C_m}^{(X)}$ and $\tau' = \tau_{C_m}^{(X')}$ with $m = 2, 4$ and they emanate from adjacent vertices *i.e.* $v_X = v_j^i$ and $v_{X'} = v_{j \pm 1}^i$. In this case, the transition function is given by $(s_0, s_h) \mapsto (s_0^{-1}, s_h)$.

Case (ii) is needed for when the emanating edges of the tropical cycle meet at the origin. For case (i), around a singular vertex v_X , we have

$$d'_0(\beta)_{\tau_{C_1}, \tau_{C_4}} = \frac{\sigma_{\tau_{C_1}}|_{S_{\tau_{C_1}, \tau_{C_4}}}}{\sigma_{\tau_{C_4}}|_{S_{\tau_{C_1}, \tau_{C_4}}}}$$

which sends

$$s_h^{(1)} \mapsto \frac{\sigma_{\tau_{C_1}}(s_h^{(1)})}{\sigma_{\tau_{C_4}}\left(\frac{1}{\mu_X s_0^{(4)}}\right)} = \mu_X, \quad s_0^{(1)} \mapsto \frac{\sigma_{\tau_{C_1}}(s_0^{(1)})}{\sigma_{\tau_{C_4}}(s_h^{(4)})} = 1.$$

Similarly, we have

$$d'_0(\beta)_{\tau_{C_2}, \tau_{C_1}} = \frac{\sigma_{\tau_{C_2}}|_{S_{\tau_{C_2}, \tau_{C_1}}}}{\sigma_{\tau_{C_1}}|_{S_{\tau_{C_2}, \tau_{C_1}}}}$$

given by

$$s_h^{(1)} \mapsto \frac{\sigma_{\tau_{C_2}}\left(\frac{s_0^{(2)}}{\mu_X}\right)}{\sigma_{\tau_{C_1}}(s_h^{(1)})} = \mu_X^{-1}, \quad s_0^{(1)} \mapsto \frac{\sigma_{\tau_{C_2}}\left(\frac{1}{s_h^{(2)}}\right)}{\sigma_{\tau_{C_1}}(s_0^{(1)})} = 1.$$

By the same argument,

$$d'_0(\beta)_{\tau_{C_4}, \tau_{C_3}} = 1 \text{ and } d'_0(\beta)_{\tau_{C_3}, \tau_{C_2}} = 1.$$

For (ii), we apply exactly the same reasoning to conclude that if τ and τ' have adjacent vertices then $d'_0(\beta)_{\tau, \tau'} = 1$.

Let $\phi_{\lambda,c} \in \text{Hom}(\text{Aff}_K, K^\times)$ denote the image of $\lambda \otimes c \in \Lambda \otimes K^\times$ under the morphism given in (9). We then have the following result:

Corollary 6.5. The lift γ of $d'_0(\beta)$ to $\prod_{\tau,\tau'} \Gamma(S_{\tau,\tau'}, \Lambda \otimes K^\times)$ is given by

$$(\check{\gamma}_{\tau,\tau'})_{\tau,\tau'} = \begin{cases} \phi_{\mathbf{e}_h^{(1)}, \mu_X} & \text{if } (\tau, \tau') = (\tau_{C_1}^{(X)}, \tau_{C_4}^{(X)}) \\ \phi_{\mathbf{e}_h^{(1)}, \mu_X^{-1}} & \text{if } (\tau, \tau') = (\tau_{C_1}^{(X)}, \tau_{C_2}^{(X)}) \\ 1 & \text{otherwise.} \end{cases}$$

Since all finite intersections of U_τ are contractible and $\Lambda \otimes K^\times$ is constant on U_τ , we have $H^i(U_{\tau_0, \dots, \tau_k}, \Lambda \otimes K^\times) = 0$. Thus by Lemma ??, we have explicitly described the representative of the K -affine structure in $\check{H}^1(\text{Sk}(U)^{\text{sm}}, \Lambda \otimes K^\times) = H^1(\text{Sk}(U)^{\text{sm}}, \Lambda \otimes K^\times)$.

6.4. Local tropical cycles. In this section, we will describe the pairing between a local tropical cycle *i.e.* a single loop surrounding a focus-focus singularity with trailing edge depicted below, and the K -affine structure we described in the previous section.

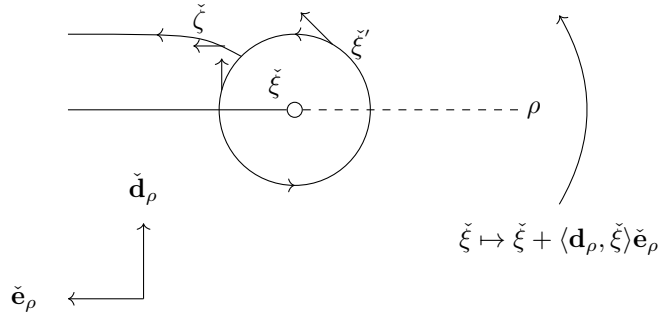
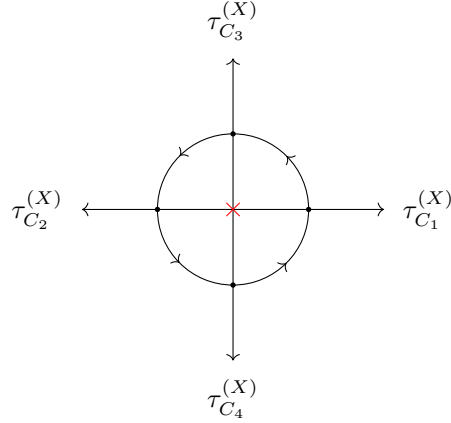


FIGURE 18. A local tropical cycle surrounding a focus-focus singularity.

We first describe a local tropical cycle with respect to the open cover above. For each loop around a non-toric vertex v_X , the tropical cycle consists of four edges between $\tau_{C_m}^{(X)}$ and $\tau_{C_{m+1}}^{(X)}$. The vertices of the edges will lie on the 1-dimensional strata $\tau_{C_m}^{(X)}$. We orient the underlying graph of the tropical cycle in the anticlockwise direction around the vertex v_X .



There are two choices of decoration (up to sign) of the graph underlying the tropical cycle. The first is given by equipping each edge with the local invariant cotangent direction, as computed in §3.8. Note that in this case, the leading edge is decorated with the zero vector, and can be discarded. The other decoration is depicted in Figure 18 where the leading edge away from the singularity is decorated with a primitive invariant cotangent direction by Lemma 4.29. For $X = Y_j^i$, by Corollary 3.47, the invariant cotangent direction is parallel to Λ_{ρ_i} .

Definition 6.6. Let $\check{\gamma}$ be a positively oriented local tropical cycle around the vertex v_X . Then $\check{\gamma}$ is either a local *invariant cycle* or a local *twisted cycle* as depicted in Figure 19 on the left and right respectively.

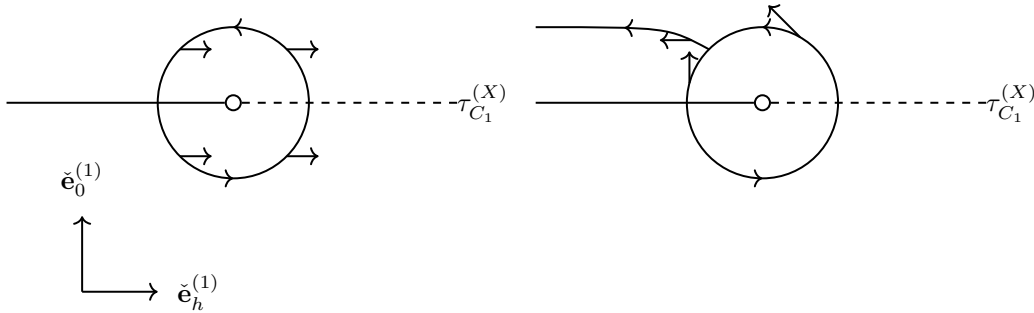


FIGURE 19. Local tropical cycles.

Proposition 6.7. Let $\check{\gamma}$ be a local positively oriented tropical cycle surrounding the vertex v_X . Then the pairing of $\check{\gamma}$ with the Čech cocycle representing the K -affine structure is

$$\langle \check{\gamma}, [\text{Aff}_K] \rangle = \begin{cases} 1 & \text{if } \gamma \text{ is an invariant cycle} \\ \mu_X & \text{if } \gamma \text{ is a twisted cycle} \end{cases}$$

where $\langle \cdot, \cdot \rangle : \check{\Lambda} \otimes (\Lambda \otimes K^\times) \rightarrow K^\times$.

Proof. Corollary 6.5 gives a description of the representative of the K -affine structure on each intersection $S_{\tau, \tau'}$. In particular, the only sections which will contribute non-trivially to the pairing will be $(\tau, \tau') = (\tau_{C_1}^{(X)}, \tau_{C_4}^{(X)})$ or $(\tau_{C_1}^{(X)}, \tau_{C_2}^{(X)})$.

If γ is an invariant cycle, then the cotangent vector decorating it is given by the invariant direction as depicted in Figure 19. By the integral affine structure computation in §3.8, the invariant cotangent direction is parallel to $\check{\mathbf{e}}_h^{(1)}$; assume $\check{\gamma}$ is equipped with the cotangent vector $\check{\mathbf{e}}_h^{(1)}$. Since the K -affine structure is trivial on $S_{\tau_{C_2}} \cap S_{\tau_{C_3}}$ and $S_{\tau_{C_3}} \cap S_{\tau_{C_4}}$, it does not contribute to the pairing. For $(\tau, \tau') = (\tau_{C_1}^{(X)}, \tau_{C_4}^{(X)})$, we have $\gamma_{\tau, \tau'} = \phi_{\check{\mathbf{e}}_h^{(1)}, \mu_X}$ and thus the pairing of $\check{\gamma}$ with $\gamma_{\tau, \tau'}$ is μ_X . Similarly, for $(\tau, \tau') = (\tau_{C_1}^{(X)}, \tau_{C_2}^{(X)})$, we have $\gamma_{\tau, \tau'} = \phi_{\check{\mathbf{e}}_h^{(1)}, \mu_X^{-1}}$ and thus the pairing of $\check{\gamma}$ with $\gamma_{\tau, \tau'}$ is μ_X^{-1} . Putting this together, the pairing of the K -affine structure with the local invariant cycle $\check{\gamma}$ is $\mu_X \cdot \mu_X^{-1} = 1$.

If γ is a twisted cycle as depicted above, then the cotangent vector decorating $\check{\gamma}$ on $U_{\tau, \tau'}$ is $\check{\mathbf{e}}_0^{(1)}$ if $(\tau, \tau') = (\tau_{C_1}^{(X)}, \tau_{C_4}^{(X)})$ and $-\check{\mathbf{e}}_h^{(1)} + \check{\mathbf{e}}_0^{(1)}$ if $(\tau, \tau') = (\tau_{C_1}^{(X)}, \tau_{C_2}^{(X)})$. Thus the pairing with the K -affine structure is $(\mu_X^{-1})^{-1} = \mu_X$ from the contribution on $S_{\tau_{C_1}^{(X)}, \tau_{C_2}^{(X)}}$. \square

6.5. Non-archimedean period map. In this section, we will use the description of the local pairing of the K -affine structure with local tropical cycles to completely describe the non-archimedean period map.

Theorem 6.8. Let

$$\text{Sk}(U)^{\text{sing}} := \{v_X : X \text{ is a non-toric irreducible component of } \mathcal{Y}_k\}$$

be the set of singular vertices in $\text{Sk}(U)$. Then the non-archimedean period map

$$\mathcal{P}^{\text{an}} : H_1(\text{Sk}(U)^{\text{sm}}, \check{\Lambda}) \longrightarrow K^\times,$$

giving by the pairing with $[\text{Aff}_K]$ is as follows:

- (1) if $\check{\gamma}$ is an invariant cycle, then $\mathcal{P}^{\text{an}} = 1$;
- (2) if $\check{\gamma}$ is an exceptional tropical cycle which surrounds v_X with positive orientation and $v_{X'}$ with negative orientation, then $\mathcal{P}^{\text{an}}(\check{\gamma}) = \frac{\mu_X}{\mu_{X'}}$;
- (3) if $\check{\gamma}$ is a spoked tropical cycle which consists of a ϵ_X -oriented loop around v_X with $X \in S \subseteq \text{Sk}(U)^{\text{sing}}$, then

$$\mathcal{P}^{\text{an}}(\check{\gamma}) = \prod_{X \in S} (\mu_X)^{\epsilon_X}.$$

Proof. This follows immediately from Proposition 6.7 where we described the pairing for local tropical cycles and noting that the emanating edges do not contribute to the pairing as they are decorated with the invariant cotangent vector. \square

This implies the following corollary:

Corollary 6.9. The non-archimedean period map \mathcal{P}^{an} factors through $H_1(\text{Sk}(U), \iota_* \check{\Lambda})$.

6.6. Comparison between algebraic periods and non-archimedean periods. In this section, we complete the proof of Theorem 1.1, which says that the non-archimedean period map agrees with the algebraic period map for generic log Calabi-Yau surfaces.

We recall the definition of the algebraic period map

$$\begin{aligned} \mathcal{P} : D^\perp &\longrightarrow \text{Pic}^0(D) \stackrel{3.18}{\cong} \mathbb{G}_m \\ L &\longmapsto L|_D. \end{aligned}$$

The line bundle $L|_D$ is of multidegree $(0, \dots, 0)$ and we can write the divisor ℓ_D associated to $L|_D$ as $\ell_D = \sum_{i=1}^n \sum_{j=1}^{r_i} (q_{ij} - p_{ij})$ with $q_{ij}, p_{ij} \in D_i^\circ$. By the isomorphism $\psi : \text{Pic}^0(D) \cong \mathbb{G}_m$ constructed in Lemma 3.18, we can rewrite this as

$$\ell_D = \sum_{i \in I} \epsilon_i (q_{j_i}^i - p_{j_i}^i)$$

with $q_{j_i}^i, p_{j_i}^i \in D_i^\circ$, where $q_{j_i}^i$ is the centre of a non-toric blowup on D_i corresponding to the non-toric exceptional curve $E_{j_i}^i$, $I \subset \{1, \dots, \ell(D)\}$ and $\epsilon_i \in \mathbb{Z}$.

Suppose $\ell_D \equiv \ell'_D := \sum_{i \in I} \epsilon_i (q_{j_i}^i - \bar{p}_{j_i}^i)$. By Lemma 3.18 we have $\prod_i z_i (p_{j_i}^i)^{\epsilon_i} = \prod_i z_i (\bar{p}_{j_i}^i)^{\epsilon_i}$ where z_i is an affine coordinate on D_i . Thus for each i we can fix a $p_i \in D_i^\circ$ which is not a centre of a non-toric blowup on D_i and write $\ell_D = \sum_{i \in I} \epsilon_i (q_{j_i}^i - p_i)$. From now on, we choose z_i such that it is compatible with the choice of coordinates we made in §5.4 *i.e.* $z_i(q_j^i) = \mu_j^i$ and we have $z_i(p_i) = 1$.

If L corresponds to an exceptional tropical cycle, then we can write $L = \mathcal{O}(E_j^i - E_k^i)$ and $\ell_D = q_j^i - q_k^i$.

We can now prove the main result of this thesis.

Theorem 6.10. The non-archimedean period map $\mathcal{P}^{\text{an}} : H_1(\text{Sk}(U), i_*\Lambda) \longrightarrow K^\times$ agrees with the algebraic period map $\mathcal{P} : D^\perp \longrightarrow K^\times$ *i.e.* the following diagram commutes.

$$\begin{array}{ccc} H_1(\text{Sk}(U), i_*\Lambda) & \xrightarrow{\mathcal{P}^{\text{an}}} & K^\times \\ & \searrow \sim & \uparrow \mathcal{P} \\ & & D^\perp \end{array}$$

Proof. Let $\tilde{\gamma}$ be a spoked tropical cycle and $L_{\tilde{\gamma}}$ the corresponding element in D^\perp . Then by the description above, we can assume $L_{\tilde{\gamma}}|_D = \mathcal{O}_D(\sum_{i \in I} \epsilon_i (q_{j_i}^i - p_i))$. We then have

$$\mathcal{P}(L_{\tilde{\gamma}}) = \prod_{i \in I} \left(\frac{z_i(q_{j_i}^i)}{z_i(p_i)} \right)^{\epsilon_i} = \prod_i (\mu_{j_i}^i)^{\epsilon_i}.$$

By Remark 4.38, we have that $\tilde{\gamma}$ has a ϵ_i -oriented local twisted tropical cycle around $v_{j_i}^i$. Thus $\mathcal{P}^{\text{an}}(\tilde{\gamma}) = \mathcal{P}(L_{\tilde{\gamma}})$.

Let $\tilde{\gamma}$ be an exceptional tropical cycle and $L_{\tilde{\gamma}}$ the corresponding element in D^\perp . We can assume $L_{\tilde{\gamma}}|_D = q_j^i - q_k^i$ and thus

$$\mathcal{P}(L_{\tilde{\gamma}}) = \frac{z_i(q_j^i)}{z_i(q_k^i)} = \frac{\mu_j^i}{\mu_k^i},$$

which agrees with $\mathcal{P}^{\text{an}}(\tilde{\gamma})$ since $\tilde{\gamma}$ has a positively oriented local twisted cycle around v_j^i and a negatively oriented local twisted cycle around v_k^i by Remark 4.38. Having checked the diagram on generators, we conclude the diagram commutes. This completes the proof of the theorem. \square

The following result realises the motivating theme of this thesis: that the essential skeleton and a K -affine structure on it remember a great deal of geometric information.

Corollary 6.11. Let (Y, D) be a generic Looijenga pair satisfying Assumption 3.22. Then the K -affine structure Aff_K on $\text{Sk}(U)$ determines the isomorphism type of U .

Proof. By Theorem 6.10, the non-archimedean period map computes the period point $D^\perp \rightarrow K^\times$. This determines an element of the moduli stack of Looijenga pairs with a marking of $\text{Pic}(Y)$ by [GHK15b, Proposition 2.9 + §6]. In particular, it determines the isomorphism type of $U = Y \setminus D$. \square

6.7. Analytic continuation to the non-generic case. In this section, we seek to remove the assumption of genericity on the Looijenga pair (Y, D) .

Definition 6.12. Let $\pi : (\mathbf{Y}, \mathbf{D}) \rightarrow S$ be a family of Looijenga pairs over a reduced connected space S . A *marking* of the family π is an isomorphism $R^2\pi_*\underline{\mathbb{Z}} \cong \underline{\Upsilon}$ where Υ is a lattice isomorphic to $H^2(\mathbf{Y}_s, \mathbb{Z})$ for some $s \in S$ and $\underline{\Upsilon}$ is the constant local system on S with fibre Υ . A marking exists if and only if the monodromy homomorphism of the family is trivial. Such families will be referred to as *marked*.

We have the following proposition concerning uniqueness of analytic continuation.

Proposition 6.13 ([ABB18] Prop. 4.2). Let X be a smooth connected strictly K -analytic space and Z a non-empty affinoid subdomain of X . Then the restriction map $\mathcal{O}(X) \rightarrow \mathcal{O}(Z)$ is injective.

Let $(\overline{Y}, \overline{D})$ be a toric Looijenga pair. Let $N > 0$ and $(k_1, \dots, k_{\ell(D)}) \in \mathbb{Z}_{\geq 0}^{\ell(D)}$ be a partition of $N > 0$ i.e. $\sum_i k_i = N$. Let $\mathbb{G}_{m, \circ}^n \subset \mathbb{G}_m^n$ denote the elements $(x_i) \in \mathbb{G}_m^n$ such that $|x_i|_K \leq 1$. We can then define the marked family of Looijenga pairs

$$\pi : (\mathbf{Y}, \mathbf{D}) \rightarrow \prod_{i=1}^{\ell(D)} \mathbb{G}_{m, \circ}^{k_i} = \mathbb{G}_{m, \circ}^N$$

such that the fibre of $(p_{ij}) \in \prod_{i=1}^{\ell(D)} \prod_{j=1}^{k_i} \mathbb{G}_{m, \circ}$ is given by the Looijenga pair arising from non-toric blowups of $(\overline{Y}, \overline{D})$ at $p_{ij} \in \overline{D}_i^\circ$. Fixing a basis for Υ , we have an isomorphism $\text{Hom}(\Upsilon, K^\times) \cong (K^\times)^r$ where $r = \text{rank}(\Upsilon)$. Define the following closed subvariety of $\mathbb{G}_{m, \circ}^N$:

$$\Delta := \{(x_{ij}) \in \prod_{i=1}^{\ell(D)} \prod_{j=1}^{k_i} \mathbb{G}_{m, \circ} : \overline{x}_{ij} = \overline{x}_{ik}\}$$

where \overline{x}_{ij} denotes the reduction of x_{ij} in $\mathcal{O}_K/(t)$. This set is precisely the points (p_{ij}) for which the specialisations coincide; this also includes the case of infinitely near points i.e. when we perform a non-toric blowup at the same point more than once and thus the generic fibre is no longer a generic Looijenga pair.

The non-archimedean period map is defined on the generic locus of this family *i.e.* away from Δ . The map can be extended naturally to all of $\mathbb{G}_{m,\circ}^N$ by the formula given in Theorem 6.8. We thus have a map

$$\mathbb{G}_{m,\circ}^N \longrightarrow (K^\times)^r$$

which we can analytify to produce

$$\nu : (\mathbb{G}_{m,\circ}^N)^{\text{an}} \longrightarrow (K^\times)^r.$$

The composition with projection onto each factor of $(K^\times)^r$ is analytic and thus ν is analytic. The non-archimedean period map has been defined on $(\mathbb{G}_{m,\circ}^N)^{\text{an}} \setminus \Delta^{\text{an}}$ and by Proposition 6.13, any extension to $(\mathbb{G}_{m,\circ}^N)^{\text{an}}$ is unique. Thus the non-archimedean period can be uniquely extended to cover the non-generic case. As a consequence, we have the following theorem:

Theorem 6.14. Let $(\mathcal{Y}, \mathcal{D})$ be a Looijenga pair over R with good reduction such that the combinatorial type of the generic fibre and special fibre agree. Then the non-archimedean period map for $U = (\mathcal{Y} \setminus \mathcal{D})_K$ agrees with the algebraic period map for U .

Proof. Given a generalised exceptional curve $C_1 + \dots + C_{k-1} + E$ arising from blowing up infinitely near points on D , we have $C_i \in D^\perp$ for $1 \leq i \leq k-1$. Since C_i is disjoint from D , we have that $\mathcal{P}(C_i) = 1$. This agrees with the non-archimedean period map and thus the proof is complete. \square

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