# Étale Homotopy Study Group

Artin and Mazur also used ideas from homotopy theory to develop the field of étale homotopy in [AM69]. The étale homotopy type of a variety is a "topological space" (sort of), whose homotopy groups give us information about the étale site of the variety: naturally, the first homotopy group is the étale fundamental group. The theory is largely based on homotopy of simplicial sets, and draws heavily from ideas in algebraic topology, and is also a good starting point for people looking to learn about motivic homotopy in the future.

In this study group, we'll define the étale homotopy type, before looking at an application to the problem of finding rational points on varieties. In [HS13], the authors define the "étale homotopy obstruction to rational points". Let $k$ be a number field, $\mathbb{A}_k$ the ring of adeles of $k$, and $X$ a smooth variety over $k$ which is a set, $X(\mathbb{A}_k)^h$ such that $X(k) \subseteq X(\mathbb{A}_k)^h \subseteq X(\mathbb{A}_k)$.

A fantastic, quick reference for the first few weeks is [SS13], though there are others that have a lot more detail, and as a result, [SS13] is a very dense paper.

## 1 Preliminary Schedule

The first three topics are pretty much fixed, as to do anything useful involving étale homotopy, you have to define the étale homotopy type. The parts about simplicial homotopy may also be interesting to topologists: it's a way of doing homotopy just by looking at sets. It will also be category theory heavy and quite dry, but I don't think there's a way around this.

I've been taking a look at some papers about étale homotopy, and the paper by Harpaz and Schlank feels like one of the more accessible papers about étale homotopy. The Harpaz and Schlank paper, [HS13] is similar to a paper by Ambrus Pál from the same year, [Pál15] which is an excellent auxillary reference, but contains more results, and is much denser.

The Harpaz and Schlank paper also contains a lot of information on the étale homology obstruction, which we won't have time to cover in this study group. As far as I am aware, sections 6,7, 8 and 10 almost all about the étale homology obstruction. Therefore, we will omit them for now.

The page numbers in italics are for [HS13].

1. **Introduction to obstructions to rational points** - The Harpaz and Schlank paper we are aiming to study talks about relating a new obstruction to rational points, the étale homotopy obstruction, to an older one,

the étale Brauer obstruction. In this session, we'll cover what we mean by the "étale Brauer obstruction to rational points".

2. **Simplicial Homotopy 1 - Model categories, simplicial sets -** Étale homotopy is based off of simplicial homotopy theory. The theory is probably too big to fit into one session, so it might be worth breaking it up into two sections: the first is roughly based on pages 1-5 of [SS13].

3. **Simplicial Homotopy 2 - Skeleton/Coskeleton, homotopy of simplicial sets -** Continuing from the week before, this should cover pages 5-9 of [SS13], and covers a simplicial version of other ideas from topology, such as homotopy and Postnikov towers.

4. **The Étale Homotopy type -** Hypercoverings, the homotopy category of hypercoverings, the definition of the étale homotopy type. If there's time, might want to see if you can prove whether the $\pi_1$ we get this way corresponds to the one defined in `https://www.jmilne.org/math/CourseNotes/LEC.pdf`. This is covered in [SS13] but is also covered in [HS13] from *pages 7-12*.

5. **A relative étale homotopy type -** Defining the relative étale homotopy type. This is quite similar to the week before, but it has some more nuance and might be worth reinforcing the ideas from before. *Pages 12-15*

6. **Homotopy Fixed Points -** Defining the set of "homotopy fixed points" of a variety. For this week, it seems like [Pál15] has a cleaner definition of homotopy fixed points, which is worth bearing in mind, but the results from [HS13] are necessary. This section is quite technical, but is worth spending time on, as the ideas in the theorems seem to be an important method of proof later on. *Pages 21-24*

7. **$p$-adic Homotopy Fixed Points, Adelic homotopy fixed points, and the étale homotopy obstruction -** This section has a few more awkward theorems that are again worth spending time on, as this forms the backbone of the paper. *Pages 25-30.*

8. **Homotopy fixed points theorems for profinite groups 1 -** The proof of lemma 4.1 and 4.2 in [HS13], as well as the statement of theorem 4.3, and the remarks, up to the point where we show that we can drop the assumption that we have a fixed point (the paragraph before "**obstruction theory**" appears). These lectures will be quite technical. *Pages 32-38*

9. **Homotopy Fixed points theorems for profinite groups 2 -** The content up to the end of corollary 4.13. This will again be very technical. *Pages 38-42*

10. **Homotopy Fixed points theorems for profinite groups 3 -** The content up to stating corollary 4.18. Again, quite technical, and builds off the previous 2 weeks. *Pages 43-46*

11. **"An auxiliary theorem"** - Chapter 5 of the text. This builds on the previous few weeks heavily, but proves a very satisfying theorem at the end: the étale homotopy obstruction depends only on the 2 truncation of the étale homotopy type. *Pages 46-51.*

12. **Connection to finite descent 1** - Chapter 9 of the text. Because this is far off, I haven't worked out a great way to split it yet, but it should be split into 2 lectures. I think a way of doing this that makes sense is for the first week to cover pages 70-72, not stating theorem 9.3, but covering lemmas 9.7 and 9.8. *Pages 70-72, and a bit more*

13. **Connection to finite descent 2** This week, we should cover theorem 9.3, which is split up into proposition 9.6 and 9.9. Whoever is taking these two sessions should co-ordinate with the other *Pages 73-79, with some exceptions*

14. **The étale Brauer obstruction is stronger than the étale homotopy obstruction** This is the beginning of the proof of the main theorem, that is, this is proposition 11.4. This first lecture will prove one half of the main theorem (namely: proposition 11.4) *Pages 99-102*

15. **The étale homotopy obstruction is stronger than étale Brauer 1** This lecture is going through the first half of the proof of proposition 11.6, all of the content up to the statement of proposition 11.9. *Pages 103-105*

16. **The étale homotopy obstruction is stronger than étale Brauer 2** The final part of proposition 11.6: we finish the proof, by proving propositions 11.9, 11.10 and 11.11. If we're not all sick of this by now, there is a chapter about applications. *Pages 106-110*

# References


[AM69] M. Artin and B. Mazur. *Etale Homotopy*. Lecture notes in mathematics. Springer, 1969.

[HS13] Yonatan Harpaz and Tomer Schlank. Homotopy obstructions to rational points. *arXiv: Algebraic Geometry*, 2013.

[Pál15] Ambrus Pál. Étale homotopy equivalence of rational points on algebraic varieties. *Algebra and Number Theory*, 9:815–873, 2015.

[SS13] T. M. Schlank and A. N. Skorobogatov. *A very brief introduction to étale homotopy*, page 61–74. London Mathematical Society Lecture Note Series. Cambridge University Press, 2013.